

Properties of Codes in the Johnson Scheme

Research Thesis

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Abstract

Codes which attain the sphere packing bound are called perfect codes. Perfect codes always draw the attention of coding theoreticians and mathematicians. The most important metrics in coding theory on which perfect codes are defined are the Hamming metric and the Johnson metric. While for the Hamming metric all perfect codes over finite fields are known, in the Johnson metric it was conjectured by Delsarte in 1970's that there are no nontrivial perfect codes. The general nonexistence proof still remains the open problem.

Constant weight codes play an important role in various areas of coding theory. They serve as building blocks for general codes in the Hamming metric. One of the applications of constant weight codes is for obtaining bounds on the sizes of unrestricted codes. In the same way as constant weight codes play a role in obtaining bounds on the sizes of unrestricted codes, doubly constant weight codes play an important role in obtaining bounds on the sizes of constant weight codes .

In this work we examine constant weight codes as well as doubly constant weight codes, and reduce the range of parameters in which perfect codes may exist in both cases.

We start with the constant weight codes. We introduce an improvement of Roos' bound for 1-perfect codes, and present some new divisibility conditions, which are based on the connection between perfect codes in Johnson graph $J(n, w)$ and block designs. Next, we consider binomial moments for perfect codes. We show which parameters can be excluded for 1-perfect codes. We examine 2-perfect codes in $J(2w, w)$ and present necessary conditions for existence of such codes. We prove that there are no 2-perfect codes in $J(2w, w)$ with length less than $2.5 * 10^{15}$.

Next we examine perfect doubly constant weight codes. We present properties of such codes, that are similar to the properties of perfect codes in Johnson graph. We present a family of parameters for codes whose size of sphere divides the size of whole space. We then prove a bound on length of such codes, similarly to Roos' bound for perfect codes in Johnson graph.

Finally we describe Steiner systems and doubly Steiner systems, which are strongly

connected with the constant weight and doubly constant weight codes respectively. We provide an anticode-based proof of a bound on length of Steiner system, prove that doubly Steiner system is a diameter perfect code and present a bound on length of doubly Steiner system.

List of symbols and abbreviations

$\binom{n}{k}$	binomial coefficient
$S(r, v)$	Stirling number of the second kind
$GF(q)$	Galois field of q elements
N	set of coordinates
n	code length
w	code weight
d	code minimum distance
e	radius
C	code
$J(n, w)$	Johnson graph
$\Phi_e(n, w)$	size of a sphere of radius e in $J(n, w)$
$t - (n, w, \lambda)$	t -design over n elements and blocks of size w
$S(t, w, n)$	Steiner system over n elements and blocks of size w
φ	code strength
$S(t_1, t_2, w_1, w_2, n_1, n_2)$	doubly Steiner system
$\Phi_e(n_1, n_2, w_1, w_2)$	size of a sphere of radius e in doubly constant code

Chapter 1

Introduction

Codes which attain the sphere packing bound are called perfect code. Perfect codes always draw the attention of coding theoreticians and mathematicians. The most important metrics in coding theory on which perfect codes are defined are the Hamming metric and the Johnson metric.

In the Hamming metric, all perfect codes over finite fields are known [1]. They exist for only a small number of parameters, while for other parameters their non-existence was proved [2, 3, 4, 1]. The nonexistence proof is based on Lloyd's polynomials. No nontrivial perfect code is known over other alphabets and for most parameters it was proved that they do not exist [5].

As for the Johnson metric, it was conjectured by Delsarte [6] in 1973 that there are no nontrivial perfect codes. Many attempts were made during the last 35 years to prove this conjecture. These attempts used Lloyd polynomials, anticode, designs and number theory. However, the previous research yielded only partial results and the general nonexistence is yet to be proved.

Perfect codes in the Johnson metric have a strong connection to constant weight codes.

Constant weight codes play an important role in various areas of coding theory. One of their applications is in obtaining lower and upper bounds on the sizes of unrestricted codes for given length and minimum Hamming distance [7, 1].

In the same way as constant weight codes are used for obtaining bounds on the sizes of unrestricted codes, doubly constant weight codes play an important role in obtaining bounds on the sizes of constant weight codes [8]. A natural question is whether there exist perfect doubly constant weight codes.

1.1 Definitions

A *binary unrestricted code* of length n is the set of binary words of length n .

The *weight* of a word is the number of ones in the word.

A *constant weight code* of length n and weight w is a binary code whose codewords have constant weight w .

A *doubly constant weight code* of length n and weight w is a constant weight code of length n and weight w , with w_1 ones in the first n_1 positions and w_2 ones in the last n_2 positions, where $n = n_1 + n_2$ and $w = w_1 + w_2$.

The *Hamming distance* (or H-distance in short) between two words of the same length n is the number of coordinates in which they differ.

If we define the distance between two words, x and y of the same weight w and the same length n , as half their H-distance, we obtain a new metric which is called the *Johnson metric* and the distance is called the *Johnson distance* (or J-distance in short).

Let $A(n, d)$ denote the maximum number of codewords in a binary code of length n and minimum H-distance d .

Let $A(n, d, w)$ denote the maximum number of codewords in a constant weight code of length n , weight w and minimum H-distance d .

A (w_1, n_1, w_2, n_2, d) code is a doubly constant weight code with w_1 ones in the first n_1 positions and w_2 ones in the last n_2 positions, and minimum J-distance d .

Let $T(w_1, n_1, w_2, n_2, \delta)$ denote the maximum number of codewords in a (w_1, n_1, w_2, n_2, d) code, where $\delta = 2d$ is a H-distance.

1.1.1 Block designs

There is a tight connection between constant weight codes and block designs.

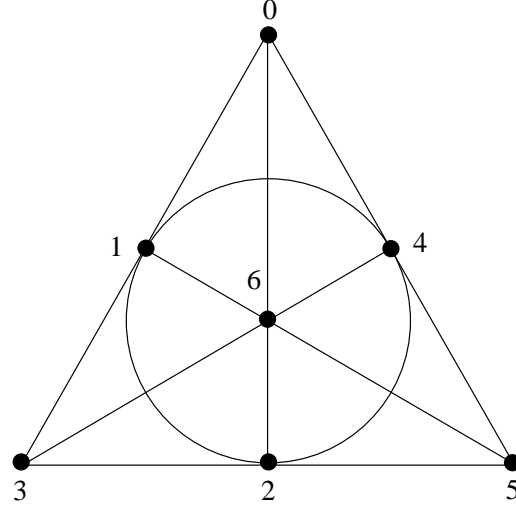
In the next chapters we will use the following terminology and properties of block designs.

Definition. Let t, n, w, λ be integers with $n > w \geq t$ and $\lambda > 0$. Let N be an n -set (i.e. a set with n elements), whose elements are called points or sometimes (for historical reasons) varieties. A $t - (n, w, \lambda)$ *design* is a collection C of distinct w -subsets called blocks of N with the property that any t -subset of N is contained in exactly λ blocks of C .

Example. If we take the lines as blocks, the seven points and seven lines (one of which is curved) of Figure 1.1 form a $2 - (7, 3, 1)$ design, since there is a unique line through any two of the seven points. The seven blocks are

$$013, 124, 235, 346, 450, 561, 602.$$

Figure 1.1: 2-(7,3,1) design



The following two theorems are well known (see [1] for reference).

Theorem 1. If $s < t$ then every t -design is also an s -design.

Notes

1. In a $t - (n, w, \lambda)$ design the total number of blocks is

$$b = \lambda \frac{\binom{n}{t}}{\binom{w}{t}}$$

2. The existence of a $t - (n, w, \lambda)$ design implies the existence of $(t - 1) - (n - 1, w - 1, \lambda)$ design (called the derived design) and $(t - 1) - (n, w, \lambda')$ design, and hence it must satisfy certain divisibility conditions:

Theorem 2. A necessary condition for a $t - (n, w, \lambda)$ design to exist, is that the numbers

$$\lambda \frac{\binom{n-i}{t-i}}{\binom{w-i}{t-i}}$$

must be integers, for $0 \leq i \leq t$.

A Steiner system is simply a t -design with $\lambda = 1$.

Definition. A Steiner system $S(t, w, n)$ is a collection of w -subsets (blocks) of n -set N such that every t -subset of N is contained in exactly one of the blocks.

Note that we use $S(t, w, n)$ as an equivalent of $t - (n, w, 1)$. Thus the example of Figure 1.1 is an $S(2, 3, 7)$.

Corollary 3. A Steiner system $S(t, w, n)$ has $\binom{n}{t} / \binom{w}{t}$ blocks.

Corollary 4. If there exists a Steiner system $S(t, w, n)$ for $t \geq 1$, then there exists a Steiner system $S(t-1, w-1, n-1)$.

Corollary 5. A necessary condition for a Steiner system $S(t, w, n)$ to exist, is that the numbers $\binom{n-i}{t-i} / \binom{w-i}{t-i}$ must be integers, for $0 \leq i \leq t$.

Incidence Matrix. Given a $t - (n, w, \lambda)$ design with n points P_1, \dots, P_n and b blocks B_1, \dots, B_b , its $b \times n$ incidence matrix $A = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } P_j \in B_i \\ 0 & \text{if } P_j \notin B_i \end{cases}$$

For example the incidence matrix of the design of Figure 1.1 is

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Codes and Designs. To every block in a $t - (n, w, \lambda)$ design corresponds a row of the incidence matrix A . If we think of these rows as codewords, the t -design forms a constant weight code C of length n and weight w .

The largest t of a code C for which the code is a t -design is called the *strength* of the code.

1.2 Perfect codes in the Hamming metric

A code C of length n and minimum H-distance $d = 2e + 1$ is called an *e-perfect* if for each vector v of length n there exists a unique element $c \in C$, such that the H-distance between v and c is at most e .

There are the *trivial* perfect codes: a code containing just one codeword, or the whole space, or a binary repetition code of odd length.

Three types of perfect codes in Hamming metric were discovered in the late 1940's:

1. The linear single-error-correcting Hamming codes $\left[n = \frac{q^m - 1}{q - 1}, n - m, 3 \right]$,
2. The binary $[23, 12, 7]$ Golay code
3. The ternary $[11, 6, 5]$ Golay code

Theorem 6 [2, 3] A nontrivial perfect code over any field $GF(q)$ must have the same parameters as one of the Hamming or Golay codes.

For non-field alphabets only trivial codes are known and it was proved that for most other parameters they do not exist. [5]

1.3 Perfect codes in the Johnson metric (survey of known results)

We associate the *Johnson graph* $J(n, w)$ with the Johnson space for given positive integers n and w such that $0 \leq w \leq n$. The vertex set V_w^n of the Johnson graph consists of all w -subsets of a fixed n -set $N = \{1, 2, \dots, n\}$. Two such w -subsets are adjacent if and only if their intersection is of size $w - 1$. A code C of such w -subsets is called an *e-perfect code* in $J(n, w)$ if the e -spheres with centers at the codewords of C form a partition of V_w^n . In other words, C is an *e-perfect code* if for each element $v \in V_w^n$ there exists a unique element $c \in C$ such that the distance between v and c is at most e .

A code C in $J(n, w)$ can be described as a collection of w -subsets of N , but it can be also described as a binary code of length n and constant weight w . From a w -subset S we construct a binary vector of length n and weight w with ones in the positions of S and zeros in the positions of $N \setminus S$. In the sequel we will use a mixed language of sets and binary vectors.

There are some *trivial perfect codes* in $J(n, w)$:

1. V_w^n is 0-perfect.
2. Any $\{v\}$, $v \in V_w^n$, $w \leq n - w$, is w -perfect.
3. If $n = 2w$, w odd, any pair of disjoint w -subsets is e -perfect with $e = \frac{1}{2}(w - 1)$.

Delsarte conjectured that there are no perfect codes in $J(n, w)$, except for these trivial perfect codes. In his seminal work from 1973 [6], he wrote :

“After having recalled that there are “very few” perfect codes in the Hamming schemes, one must say that, for $1 < \delta < n$, there is not a single one known in the Johnson schemes. It is tempting to risk the conjecture that such codes do not exist. “

Indeed, Delsarte omitted the trivial perfect codes (we will omit them too, unless otherwise stated, so when we say perfect codes we mean nontrivial perfect codes), and his conjecture on the nonexistence of perfect codes in the Johnson spaces has provided plenty of ground for research in the years which followed. Due to the fact that in the Hamming spaces over $GF(q)$ all parameters for which perfect codes exist were known, special emphasis was given to the Johnson spaces. However, not many significant results were produced.

A connected graph Γ with diameter d is called *distance-regular* if for any vertices x and y of Γ and any integers $0 \leq i, j \leq d$, the number of vertices z at distance i from x and at distance j from y depends only on i, j and $k := \text{dist}(x, y)$ and not on the choice of x and y themselves.

The following theorem is due to Delsarte [6]:

Theorem 7 : Let X and Y be subsets of the vertex set V of a distance regular graph Γ , such that nonzero distances occurring between vertex in X do not occur between vertices of Y . Then $|X| \cdot |Y| \leq |V|$.

A subset X of V is called an *anticode* with diameter D , if D is the maximum distance occurring between vertices of X .

Anticodes with diameter D having maximal size are called *optimal anticodes*.

Let Γ be a connected graph. We denote by $d_\Gamma(x, y)$ the length of the shortest path from x to y . Γ is said to be *distance transitive* if, whenever x, x', y, y' are vertices with $d_\Gamma(x, x') = d_\Gamma(y, y')$, there is an automorphism γ of Γ with $\gamma(x) = y$ and $\gamma(x') = y'$. A distance-transitive graph is obviously distance regular.

Biggs [9] showed that the natural setting for the existence problem of perfect codes is the class of distance transitive graphs. Biggs claims that the class of distance transitive graphs includes all interesting schemes, such as the Hamming scheme and the Johnson scheme, and developed a general theory and a criterion for the existence of perfect codes in a distance-transitive graph. He showed that this criterion implies Lloyd's theorem, which is used in the Hamming scheme to prove the nonexistence of perfect codes in all cases.

Bannai [10] proved the nonexistence of e -perfect codes in $J(2w - 1, w)$ and $J(2w + 1, w)$, for $e \geq 2$. He used an analogue to Lloyd's theorem and some number-theoretic results.

Hammond [11] extended this result and showed that $J(n, w)$ can not contain a non-trivial perfect code for $n \in \{2w - 2, 2w - 1, 2w + 1, 2w + 2\}$.

Theorem 8 [11]. There are no perfect codes in $J(2w-2, w)$, $J(2w-1, w)$, $J(2w+1, w)$ and $J(2w+2, w)$.

However, the most significant result, in the first twenty years following Delsarte's conjecture, was given in 1983 by Roos [12].

Theorem 9 [12]. If an e -perfect code in $J(n, w)$, $n \geq 2w$, exists, then $n \leq (w-1)\frac{2e+1}{e}$.

The proof of Roos was based on anticodes. By using Theorem 7, Roos noticed that if an e -perfect code exists, then the e -spheres should be optimal anticodes with diameter $2e$. He proceeded to find anticodes in $J(n, w)$ and obtained his result by comparing them to the e -spheres.

Etzion in [13] give a different simple proof of this theorem and in [14] Etzion and Schwartz show that no nontrivial e -perfect code achieves Roos' bound with equality.

Another approach was shown by Etzion in [15]. He proved that if there exists a non-trivial e -perfect code C in $J(n, w)$, then many Steiner systems are embedded in C . Using Etzion's approach, the necessary conditions for the existence of Steiner systems imply necessary conditions for the existence of perfect codes in the Johnson graph. Moreover, Etzion developed a new concept called configuration distribution, which is akin to the concept of weight distribution for codes in the Hamming metric. Using this concept, combined with the necessary conditions derived from Steiner systems, many parameters were found, for which e -perfect codes do not exist in $J(n, w)$. We summarize the main results given in [15, 13]:

Lemma 10. If C is an e -perfect code in the Johnson scheme then its minimum H-distance is $4e+2$.

Lemma 11. If C is an e -perfect code in the $J(n, w)$ then $A(n, 4e+2, w) = |C|$.

Let $N = \{1, 2, \dots, n\}$ be the n -set. From a Steiner system $S(t, w, n)$ we construct a constant-weight code on n coordinates as follows. From each block B we construct a codeword with ones in the positions of B and zeros in the positions of $N \setminus B$. This construction leads to the following well known theorem [16].

Theorem 12. $A(n, 2(k-t+1), k) = \frac{n(n-1)\dots(n-t+1)}{k(k-1)\dots(k-t+1)}$ if and only if a Steiner system $S(t, k, n)$ exists.

From Theorem 12 and Lemma 10 we immediately infer the following result.

Lemma 13. If C is an e -perfect code in $J(n, w)$ which is also a Steiner system, then it is a Steiner system $S(w-2e, w, n)$.

The next lemma is a simple observation of considerable use.

Lemma 14. The complement of an e -perfect code in $J(n, w)$ is an e -perfect code in $J(n, n-w)$.

If we combine Lemma 4 with the fact that the J-distance between words of an e -perfect code is at least $2e+1$, we get:

Corollary 15. If an e -perfect code exists in $J(n, w)$, then $w \geq 2e + 1$ and $n - w \geq 2e + 1$.

For a given partition of N into two subsets, A and B , such that $|A| = k$ and $|B| = n - k$, let *configuration* (i, j) consist of all vectors with weight i in the positions of A and weight j in the positions of B .

For an e -perfect code C in $J(n, w)$, we say that $u \in C$ J -covers $v \in V_w^n$ if the J -distance between u and v is less than or equal to e . For a given two subsets u and v we say that u C -covers v if v is a subset of u .

Theorem 16. If an e -perfect code exists in $J(n, w)$, then a Steiner system $S(e + 1, 2e + 1, w)$ and a Steiner system $S(e + 1, 2e + 1, n - w)$ exist.

Theorem 17. If an e -perfect code exists in $J(n, w)$, then a Steiner system $S(2, e + 2, w - e + 1)$ and a Steiner system $S(2, e + 2, n - w + e - 1)$ exist.

Corollary 18. If an e -perfect code exists in $J(n, w)$, then $n - w \equiv w \equiv e \pmod{e + 1}$ and hence $e + 1$ divides $n - 2w$.

Theorem 19. Except for the Steiner systems $S(1, w, n)$ and $S(w, w, n)$, there are no more Steiner systems which are also perfect codes in the Johnson scheme.

Theorem 20. An e -perfect code in $J(2w, w)$ is self-complement, i.e., the complement of the code is equal to the code.

Theorem 21. There are no e -perfect codes in $J(2w + p, w)$, p prime, in $J(2w + 2p, w)$, p is a prime, $p \neq 3$, and in $J(2w + 3p, w)$, p is a prime, $p \neq 2, 3, 5$.

Theorem 22. If an e -perfect code exists in $J(n, w)$ and $n < (w - 1)(2e + 1)/e$, then a $S(2, e + 2, n - w + 2)$ exists.

Corollary 23. If an e -perfect code in $J(n, w)$ exists and $w \leq n - w$, then a $S(2, e + 2, w + 2)$ exists.

Now, we consider the Steiner systems which are embedded in an e -perfect code in $J(n, w)$. By using the necessary condition for existence of Steiner system, we have the following results.

Theorem 24. Assume there exists an e -perfect code in $J(n, w)$.

- If e is odd then n is even and $(e + 1)(e + 2)$ divides $n - 2w$.
- If e is even and n is even then $(e + 1)(e + 2)$ divides $n - 2w$.
- If e is even and n is odd then $e \equiv 0 \pmod{4}$ and $\frac{(e+1)(e+2)}{2}$ divides $n - 2w$.

Corollary 25. There are no perfect codes in:

- $J(2w + p^i, w)$, p is a prime and $i \geq 1$.
- $J(2w + pq, w)$, p and q primes, $q < p$, and $p \neq 2q - 1$.

Etzion and Schwartz [14] introduced the concept of t -regular codes.

We summarize some of the relevant results from [14].

Theorem 26. If an e -perfect code C in $J(n, w)$ is t -regular, then

$$\Phi_e(n, w) \mid \binom{n-i}{w-i},$$

for all $0 \leq i \leq t$, where $\Phi_e(n, w)$ denotes the size of sphere of radius e .

Define the following polynomial :

$$\sigma_e(w, a, t) = \sum_{j=0}^e (-1)^j \binom{t}{j} \sum_{i=0}^{e-j} \binom{w-j}{i} \binom{w+a-t+j}{i+j}.$$

Theorem 27. Let C be an e -perfect code in $J(2w+a, w)$, and let $1 \leq t \leq w$. If $\sigma_e(w, a, m) \neq 0$ for all the integers $1 \leq m \leq t$, then C is t -regular.

Theorem 28. If a 1-perfect code exists in $J(2w+a, w)$, then it is t -regular for all

$$0 \leq t \leq \frac{2w+a+1 - \sqrt{(a+1)^2 + 4(w-1)}}{2}. \quad (1.1)$$

Theorem 29. There are no 1-perfect codes in $J(n, w)$, when

$$\Phi_1(n, w) = 1 + w(n-w) \equiv 0 \pmod{4}.$$

Theorem 30. If an e -perfect code, $e \geq 2$, exists in $J(2w+a, w)$, then it is t -regular for all $0 \leq t \leq \frac{w}{e} - e$.

Corollary 31. If an e -perfect code exists in $J(n, w)$, then it is e -regular.

Theorem 32. For all $e \geq 2$, there exists $W_e > 0$, such that for all $w \geq W_e$, all e -perfect codes in $J(2w+a, w)$ are $\lfloor \frac{w}{2} \rfloor$ -regular.

Theorem 33. There are no e -perfect codes in $J(n, w)$, $e \geq 2$, which are also $\lfloor \frac{w}{2} \rfloor$ -regular, when $\Phi_e(n, w) \equiv 0 \pmod{p^2}$, p a prime.

Theorem 34. Let p be a prime, and $e \equiv -1 \pmod{p^2}$. If an e -perfect code exists in $J(n, w)$, then

$$\Phi_e(n, w) \equiv 0 \pmod{p^2}.$$

Corollary 35. For any given $e \geq 2$, $e \equiv -1 \pmod{p^2}$, p a prime, there are finitely many nontrivial e -perfect codes in the Johnson graph.

Theorem 36. There are no nontrivial 3-perfect, 7-perfect, 8-perfect codes in the Johnson graph.

Martin [17] also examined the existence problem when he considered completely-regular subsets in his thesis. He found that if $e = 1$, then perfect codes must obey some numerical formula: $w = rs + 1$ and $n = 2rs + r - s + 1$. Etzion [18] has shown that these observations are implied from (1.1) .

Ahlsweide, Aydinian and Khachatrian [19] gave a new interesting definition of *diameter-perfect codes* (D-perfect codes). They examined a variant of Theorem 7(of Delsarte). Let Γ be a distance-regular graph with a vertex set V . If A is an anticode in Γ , denote by $D(A)$ the diameter of A . Now let

$$A^*(D) = \max \{|A| : D(A) \leq D\}.$$

Theorem 37. If C is a code in Γ with minimum distance $D + 1$, then $|C| \leq |V|A^*(D)^{-1}$.

They continued with the following new definition for perfect codes. A code C with minimum distance $D + 1$ is called D -perfect if Theorem 37 holds with equality. This is a generalization of the usual definition of e -perfect codes as e -spheres are anticodes with diameter $2e$.

Gordon [20] proved that size of sphere of 1-perfect code in $J(n, w)$ is squarefree, and for each prime $p_i | \Phi_1(n, w)$, there is an integer α_i such that $p_i^{\alpha_i}$ must be close to $n - w$, moreover, the α_i 's are distinct and pairwise coprime, and the sum of their reciprocals is close to two.

1.4 Organization of this work

The rest of this thesis is organized as follows.

In Chapter 2 we examine perfect codes in the Johnson graph. We start by a brief survey of the techniques concerning the existence of perfect codes in the Johnson graph, which are relevant to our work. Then we introduce the improvement of Roos bound for 1-perfect codes, and present some new divisibility conditions. Next, we consider binomial moments for perfect codes and show which general parameters can be ruled out. Finally we examine 2-perfect codes in $J(2w, w)$ and present necessary conditions for existence of such codes, using Pell equations.

In Chapter 3 we examine perfect doubly constant weight codes. We present the properties of such codes, that are similar to the properties of perfect codes in Johnson graph, construct the family of parameters for codes whose sphere divides the size of whole space and finally prove the bound on length on such codes, that is similar to Roos' bound for perfect codes in Johnson graph.

Chapter 4 deals with Steiner systems and doubly Steiner systems. We provide an anticode-based proof of the bound on Steiner system, prove that doubly Steiner system is a diameter perfect code and present the bound on the size of doubly Steiner system.

Chapter 2

Perfect codes in $J(n, w)$

2.1 t -designs and codes in $J(n, w)$

In this section we use t - designs and the strength of the code for excluding Johnson graphs in which there are no e -perfect codes. We introduce the notion of t -regular codes, and their properties, as presented in [14].

In $J(n, w)$, let

$$\Phi_e(n, w) = \sum_{i=0}^e \binom{w}{i} \binom{n-w}{i},$$

denote size of sphere of radius e . The number of codewords in an e -perfect code C in $J(n, w)$ is

$$|C| = \frac{\binom{n}{w}}{\Phi_e(n, w)}$$

by the sphere packing bound, hence

$$\Phi_e(n, w) \left| \binom{n}{w} \right|.$$

However, we learn much more about perfect codes, by using the approach which was presented in [14]. Now we introduce the definition of t -regular codes:

Definition 1. Let C be a code in $J(n, w)$ and let A be a subset of the coordinate set N . For $0 \leq i \leq |A|$ we define

$$C_A(i) = |\{c \in C : |c \cap A| = i\}|.$$

Also, for each $I \subseteq A$ we define

$$C_A(I) = |\{c \in C : c \cap A = I\}|.$$

Definition 2. A code C in $J(n, w)$ is said to be t -regular, if the following two conditions hold:

(c.1) There exist numbers $\alpha(0), \dots, \alpha(t)$ such that if $A \subset N$, $|A| = t$, then $C_A(i) = \alpha(i)$ for all $0 \leq i \leq t$.

(c.2) For any given t -subset A of N , there exist numbers $\beta_A(0), \dots, \beta_A(t)$ such that if $I \subset A$ then $C_A(I) = \beta_A(|I|)$.

Note that if a code is t -regular, $t \geq 1$, then it is also $(t-1)$ -regular.

It was proved in [18] that a code C in $J(n, w)$ is t -regular if and only if it forms t -design. The strength of an e -perfect code C can be used to exclude the existence of perfect codes by the following theorem [14].

Theorem 37. If an e -perfect code C in $J(n, w)$ is t -regular, then

$$\Phi_e(n, w) \Big| \binom{n-i}{w-i},$$

for all $0 \leq i \leq t$.

It was proved in [14] that if C is an e -perfect code in $J(n, w)$ with strength φ then

$$\sum_{i=0}^e (-1)^i \binom{\varphi+1}{i} \sum_{j=0}^{e-i} \binom{w-i}{j} \binom{n-w-\varphi-1+i}{i+j} = 0$$

and for $t \leq \varphi$

$$\sum_{i=0}^e (-1)^i \binom{t}{i} \sum_{j=0}^{e-i} \binom{w-i}{j} \binom{n-w-t+i}{i+j} \neq 0.$$

Therefore, the polynomial $\sigma_e(n, w, t) = \sum_{i=0}^e (-1)^i \binom{t}{i} \sum_{j=0}^{e-i} \binom{w-i}{j} \binom{n-w-t+i}{i+j}$, defined in [14] satisfies the following condition: the smallest positive integer φ for which $\sigma_e(n, w, \varphi+1) = 0$ is the strength of C .

When $e = 1$, $\sigma_e(n, w, t)$ is quadratic equation and φ is easily computed:

$$\varphi = \frac{n-1 - \sqrt{(n-2w+1)^2 + 4(w-1)}}{2} \quad (2.1)$$

Note, that when $e \geq 2$, $\sigma_e(n, w, t)$ is much more complicated polynomial, and it is tempting to conjecture that there are no integer solutions to $\sigma_e(n, w, t) = 0$ for $e > 2$.

2.1.1 Divisibility conditions for 1-perfect codes in $J(n, w)$

Now we prove the theorem which provides divisibility conditions for 1-perfect codes in $J(2w+a, w)$.

Theorem 38. If there exists a 1-perfect code C with strength $w - d$ for some $d \geq 0$ in $J(2w + a, w)$, then

1. $w - d \equiv 0, 1, 4 \text{ or } 9 \pmod{12}$
2. $\lambda := \frac{\prod_{i=0}^{d-2} (wd - (d+i(d-1)))}{(d-1)!(d-1)^{d-1}d(w-d+1)} \in \mathbb{Z}$
3. $\lambda \prod_{j=1}^s \left\lceil \frac{wd+jd-(j+1)}{(d-1)(d+j)} \right\rceil \in \mathbb{Z}, 0 \leq s \leq w - d$

Proof. Assume that there exists a 1-perfect code in $J(2w + a, w)$. Therefore, by (2.1), the strength of C is

$$\frac{2w + a - 1 - \sqrt{(a+1)^2 + 4(w-1)}}{2}.$$

Define the following function of w and a

$$f(w, a) = \frac{2w + a - 1 - \sqrt{(a+1)^2 + 4(w-1)}}{2}.$$

Note that $f(w, a)$ is an increasing function of a .

Now suppose that $f(w, a) = w - d$. Therefore, we get the following expression for a :

$$a = \frac{w - d^2 + d - 1}{d - 1},$$

therefore, $d > 1$.

Now we use the following lemma [14]:

Lemma 39. If there exists a 1-perfect code in $J(n, w)$ then either $w \equiv n - w \equiv 1 \pmod{12}$ or $w \equiv n - w \equiv 7 \pmod{12}$.

In particular, $w \equiv 1 \pmod{6}$ and $6 \mid a$, hence given that $w = 6k + 1$ for some integer k , it follows that

$$6 \mid a = \frac{6k - d^2 + d}{d - 1}$$

or

$$6 \mid d^2 - d.$$

Therefore, $d \equiv 0 \pmod{3}$ or $d \equiv 1 \pmod{3}$. We write this result modulo 12: $d \equiv 0, 1, 3, 4, 6, 7, 9 \text{ or } 10 \pmod{12}$.

Now we consider all the values of d modulo 12 and relate them to the values of w and $w - d$, e.g. the strength, modulo 12.

Since $a = \frac{w-d^2+d-1}{d-1}$,

$$w = (d-1)a + d^2 - d + 1$$

From Lemma 39, $12|a$, thus

$$w \equiv d^2 - d + 1 \pmod{12}.$$

1. $d \equiv 0 \pmod{12}$: $w \equiv 1 \pmod{12}$, $w - d \equiv 1 \pmod{12}$.
2. $d \equiv 1 \pmod{12}$: $w \equiv 1 \pmod{12}$, $w - d \equiv 0 \pmod{12}$.
3. $d \equiv 3 \pmod{12}$: $w \equiv 7 \pmod{12}$, $w - d \equiv 4 \pmod{12}$.
4. $d \equiv 4 \pmod{12}$: $w \equiv 1 \pmod{12}$, $w - d \equiv 9 \pmod{12}$.
5. $d \equiv 6 \pmod{12}$: $w \equiv 7 \pmod{12}$, $w - d \equiv 1 \pmod{12}$.
6. $d \equiv 7 \pmod{12}$: $w \equiv 7 \pmod{12}$, $w - d \equiv 0 \pmod{12}$.
7. $d \equiv 9 \pmod{12}$: $w \equiv 1 \pmod{12}$, $w - d \equiv 4 \pmod{12}$.
8. $d \equiv 10 \pmod{12}$: $w \equiv 7 \pmod{12}$, $w - d \equiv 9 \pmod{12}$.

This proves the first part of the theorem.

Now we will find the divisibility conditions of the second and the third parts of the theorem.

Note that by using the expression for a :

$$a = \frac{w-d^2+d-1}{d-1},$$

we can represent the size of the sphere as follows:

$$\Phi_1(w, a) = 1 + w(w+a) = (w+a+d)(w-d+1).$$

The code C is a $t - (n, w, \lambda_t)$ -design for each t , $0 \leq t \leq w-d = f(w, a)$, where

$$\lambda_t = \frac{\binom{n-t}{w-t}}{\Phi_1(n, w)} = \frac{\binom{2w+a-t}{w+a}}{(w+a+d)(w-d+1)}.$$

Let denote

$$\lambda := \lambda_{w-d} = \frac{\binom{w+a+d}{w+a}}{(w+a+d)(w-d+1)}.$$

We simplify the expression for λ , by using that $w+a+d-1 = \frac{w(d-1)+w-d^2+d-1+(d-1)^2}{d-1} = \frac{wd-d}{d-1}$:

$$\begin{aligned}\lambda &= \frac{\binom{w+a+d-1}{d-1}}{d(w-d+1)} = \frac{\binom{\frac{wd-d}{d-1}}{\frac{wd-d}{d-1}}}{d(w-d+1)} = \frac{\left(\frac{wd-d}{d-1}\right)!}{(d-1)! \left(\frac{wd-d}{d-1} - (d-1)\right)! d(w-d+1)} \\ &= \frac{\left(\frac{wd-d}{d-1}\right) \left(\frac{wd-d}{d-1} - 1\right) \dots \left(\frac{wd-d}{d-1} - (d-2)\right)}{(d-1)! d(w-d+1)} \\ &= \frac{(wd-d)(wd-d-(d-1)) \dots (wd-d-(d-2)(d-1))}{(d-1)! (d-1)^{d-1} d(w-d+1)}\end{aligned}$$

Thus we get the first divisibility condition:

$$\lambda = \frac{\prod_{i=0}^{d-2} (wd - (d+i(d-1)))}{(d-1)! (d-1)^{d-1} d(w-d+1)} \in \mathbb{Z}.$$

The code C is a $t - (n, w, \lambda_t)$ -design for each t , $0 \leq t \leq w-d = f(w, a)$, therefore for all t , $0 \leq t \leq w-d$

$$\Phi_1(w, a) \left| \binom{2w+a-t}{w-t} \right|$$

or for all $0 \leq s \leq w-d$,

$$\Phi_1(w, a) \left| \binom{w+a+d+s}{w+a} \right|.$$

Note that

$$\binom{w+a+d+s}{w+a} = \binom{w+a+d}{w+a} \frac{(w+a+d+1)(w+a+d+2) \dots (w+a+d+s)}{(d+1)(d+2) \dots (d+s)},$$

where $0 \leq s \leq w-d$.

Note also that

$$\frac{\binom{w+a+d}{w+a}}{\Phi_1(w, a)} = \lambda,$$

therefore, the last condition can be rewritten as follows:

$$\lambda \frac{(w+a+d+1)(w+a+d+2) \dots (w+a+d+s)}{(d+1)(d+2) \dots (d+s)} \in \mathbb{Z},$$

for all $0 \leq s \leq w - d$.

Since $w + a + d + s = \frac{wd + sd - (s+1)}{d-1}$ we finally get the second divisibility condition:

$$\lambda \prod_{j=1}^s \left[\frac{wd + jd - (j+1)}{(d-1)(d+j)} \right] \in \mathbb{Z},$$

for all $0 \leq s \leq w - d$, where

$$\lambda = \frac{\prod_{i=0}^{d-2} (wd - (d + i(d-1)))}{(d-1)!(d-1)^{d-1}d(w-d+1)} \in \mathbb{Z}$$

□

2.1.2 Improvement of Roos' bound for 1-perfect codes

From the Roos' bound, it follows that if a 1-perfect code exists in $J(2w + a, w)$, then

$$2w + a \leq 3(w - 1)$$

or

$$a \leq w - 3.$$

Now we use the divisibility conditions from the previous section in order to improve this bound.

Theorem 40. If a 1-perfect code exists in $J(2w + a, w)$, then

$$a < \frac{w}{11}.$$

Proof. Assume that there exists a 1-perfect code C in $J(2w + a, w)$ and that the strength of C is $w - d$. Then, by Theorem 38

$$\lambda = \frac{\prod_{i=0}^{d-2} (wd - (d + i(d-1)))}{(d-1)!(d-1)^{d-1}d(w-d+1)} \in \mathbb{Z}. \quad (2.2)$$

Given $w = 6k + 1$ for some integer k , we rewrite the expression for λ as follows:

$$\lambda = \frac{\prod_{i=0}^{d-2} (6kd - i(d-1))}{(d-1)!(d-1)^{d-1}(d(6k+1) - d^2 + d)}. \quad (2.3)$$

Since $d - 1 \mid 6k$, we rewrite the last expression as

$$\lambda = \frac{\left(d \frac{6k}{d-1}\right) \left(d \frac{6k}{d-1} - 1\right) \dots \left(d \frac{6k}{d-1} - (d-2)\right)}{(d-1)!(d(6k+1) - d^2 + d)}.$$

Note, that the numerator contains $d - 1$ successive numbers, therefore $(d - 1)!$ divides it. In addition, $d - 1$ does not divide $d(6k + 1) - d^2 + d$, because $\gcd(d - 1, d) = 1$ and $\gcd(d - 1, 6k + 1) = 1$, therefore we should determine if $d(6k + 1) - d^2 + d$ divides the numerator of (2.3), or if $d(w - d + 1)$ divides the numerator of (2.2). Note also that the size of the sphere must be squarefree [20], in particular the expression $w - d + 1$ must be squarefree as a factor of Φ_1 .

Now we examine several first values of $d > 1$.

- $d = 3$. From (2.2)

$$\lambda = \frac{(3w-3)(3w-5)}{2!2^23(w-2)} = \frac{(w-1)(3w-5)}{8(w-2)},$$

and since $\gcd(w-1, w-2) = 1$ and $\gcd(3w-5, 3w-6) = 1$, $\lambda \notin \mathbb{Z}$. Contradiction.

Therefore, $d > 3$ and $a \leq \frac{w-4^2+4-1}{3} < \frac{w}{3}$.

- $d = 4$. From (2.1)

$$\lambda = \frac{4(w-1)(4w-7)(4w-10)}{3!3^34(w-3)},$$

therefore, all possible factors of $w - 3$ are 2 and 5, but $a = \frac{w-13}{3}$, thus $w > 13$. Contradiction.

Therefore, $d > 4$ and $a \leq \frac{w-6^2+6-1}{5} < \frac{w}{5}$.

- $d = 6$. From (2.2)

$$\lambda = \frac{6(w-1)(6w-11)(6w-16)(6w-21)(6w-26)}{5!5^56(w-5)},$$

therefore, all possible factors of $w - 5$ are 2, 19, 7 and 3. But $w \equiv 1 \pmod{6}$, hence $w - 5 \equiv 2 \pmod{6}$, so $w - 5 = 2 * 7$, or $w - 5 = 2 * 19$, or $w - 5 = 2 * 7 * 19$, therefore $w = 19, 43$ or 271 . But $a = \frac{w-31}{5}$, so the only possible value for w is 271 and $a = 48$. But

it must be that $\Phi_1(n, w) \mid \binom{n-i}{w-i}$ for all $0 \leq i \leq w - 6$, and for $i = w - 7$ it is false.

Therefore, $d > 6$, and $a \leq \frac{w-7^2+7-1}{6} < \frac{w}{6}$.

- $d = 7$. From (2.2)

$$\lambda = \frac{7(w-1)(7w-13)(7w-19)(7w-25)(7w-31)(7w-37)}{6! * 6^6 * 7(w-6)},$$

therefore, all possible factors of $w-6$ are 5, 29, 23, 17 or 11. Since $w-6 \equiv 1 \pmod{6}$ and all possible factors are $-1 \pmod{6}$, the number of factors of $w-6$ is even. Note that $w-6 \equiv 1 \pmod{4}$, all factors are $\pm 1 \pmod{4}$, and only $23 \equiv -1 \pmod{4}$ and $11 \equiv -1 \pmod{4}$. Thus 23 and 11 either appear together or do not appear at all.

Given that $w = 6k + 1$, k is integer, then

$$12 \mid a = \frac{6k-42}{6} = k-7,$$

therefore,

$$k \equiv 7 \pmod{12}. \quad (2.4)$$

Thus all possible cases are:

1. 4 factors: $w-6 = 6k-5 = 23 * 11 * 5 * 29$, $23 * 11 * 5 * 17$ or $23 * 11 * 29 * 17$. In any case we obtain contradiction to (2.4), except for $w-6 = 23 * 11 * 5 * 29$, in which case $w = 36691$, $a = 6108$. Here $\Phi_1(n, w)$ does not divide $\binom{n-i}{w-i}$ for $i = w-11$.
2. 2 factors: $w-6 = 6k-5 = 23 * 11$, $5 * 29$, $5 * 17$ or $29 * 17$. In any case we obtain contradiction to (2.4), except for $w-6 = 23 * 11$, in which case $w = 259$, $a = 36$. Here $\Phi_1(n, w)$ does not divide $\binom{n-i}{w-i}$ for $i = w-8$.

In any case we obtain contradiction, therefore, $d > 7$ and $a \leq \frac{w-9^2+9-1}{8} < \frac{w}{8}$.

- $d = 9$. From (2.2)

$$\lambda = \frac{9(w-1)(9w-17)(9w-25)(9w-33)(9w-41)(9w-49)(9w-57)(9w-65)}{8! * 8^8 * 9(w-8)},$$

therefore, all possible factors of $w-8$ are 5, 7, 11, 13, 23, 31 and 47.

Note that $12 \mid a = \frac{6k-72}{8}$, so $12 * 8 \mid 6k-72$, therefore, $k = 16b + 12$, for some integer b , $w = 96b + 73$, $w-8 \equiv 1 \pmod{16} \equiv 5 \pmod{12}$.

Note that $5 \equiv 5 \pmod{12}$, $7 \equiv 23 \equiv -5 \pmod{12}$, $11 \equiv 23 \equiv 47 \equiv -1 \pmod{12}$, $13 \equiv 1 \pmod{12}$, $23 \equiv 7 \pmod{16}$, $31 \equiv 47 \equiv -1 \pmod{16}$, $11 \equiv -5 \pmod{16}$.

Thus all possible cases are:

1. 6 factors: $w - 8 = 5 * 7 * 13 * 23 * 31 * 47$, in this case $\Phi_1(n, w)$ does not divide $\binom{n-i}{w-i}$ for $i = w - 10$. Contradiction.
2. 5 factors: $w - 8 = 5 * 7 * 11 * 31 * 47$ or $7 * 11 * 13 * 23 * 47$. In both cases $\Phi_1(n, w)$ does not divide $\binom{n-i}{w-i}$ for $i = w - 10$. Contradiction.
3. 3 factors: $w - 8 = 5 * 11 * 23$ or $11 * 13 * 31$. In both cases $\Phi_1(n, w)$ does not divide $\binom{n-i}{w-i}$ for $i = w - 11$. Contradiction.
4. 2 factors: $w - 8 = 5 * 13, 7 * 23$ or $31 * 47$, In the first case $\Phi_1(n, w)$ does not divide $\binom{n-i}{w-i}$ for $i = w - 13$, and in the last two cases $\Phi_1(n, w)$ does not divide $\binom{n-i}{w-i}$ for $i = w - 10$. Contradiction.

Therefore, $d > 9$ and $a \leq \frac{w-10^2+10-1}{9} < \frac{w}{9}$.

- $d = 10$. From (2.2)

$$\lambda = \frac{10}{9! * 9^9 * 10(w-9)} [(w-1)(10w-19)(10w-28)(10w-37)(10w-46) * (10w-55)(10w-64)(10w-73)(10w-82)],$$

therefore, all possible factors of $w - 9$ are 2, 71, 31, 53, 11, 7, 13 and 17.

Note that $12|a = \frac{w-91}{9} = \frac{12k+7-91}{9} = \frac{12k-84}{9}$, thus $12*9|12k-84$, or $9|k-7$, so we can write $k = 9b + 7$, for some integer b . Also $w - 9 \equiv 1 \pmod{9} \equiv 2 \pmod{4} \equiv -2 \pmod{6} \equiv -2 \pmod{12}$. If we consider all possible factors modulo 9, 4, 6 and 12, we get several constraints, therefore the only possible cases are:

1. 5 factors: $w - 9 = 2 * 7 * 13 * 17 * 31$, $2 * 17 * 31 * 53 * 71$, $2 * 7 * 13 * 31 * 53$ or $2 * 7 * 11 * 17 * 53$. In the two first cases 12 does not divides a . In the third case $\Phi_1(n, w)$ does not divide $\binom{n-i}{w-i}$ for $i = w - 14$. In the last case $\Phi_1(n, w)$ does not divide $\binom{n-i}{w-i}$ for $i = w - 11$. Contradiction.
2. 3 factors: $w - 9 = 2 * 7 * 11$, $2 * 13 * 17$, $2 * 13 * 53$, or $2 * 31 * 71$. In the first two cases 12 does not divide a . In the last two cases $\Phi_1(n, w)$ does not divide $\binom{n-i}{w-i}$ for $i = w - 13$. Contradiction.

Therefore, $d > 10$. Moreover, since $d \equiv 0, 1 \pmod{3}$, $d \geq 12$.

Conclusion:

$$a \leq \frac{w - 12^2 + 12 - 1}{11} = \frac{w - 133}{11} < \frac{w}{11}.$$

□

Note, that while we do not show a generalization, we can further improve the bound on a by applying this technique.

2.1.3 Number theory's constraints for size of $\Phi_1(n, w)$

In [18] Etzion shown that if 1-perfect C code exists in $J(n, w)$, then

$$w = (\beta - \alpha)(\beta + \alpha + 1) + 1,$$

$$n = 2(\beta - \alpha)(\beta + \alpha + 1) + 2\alpha + 2,$$

and the strength of the code C is

$$(\beta - \alpha)(\beta + \alpha),$$

where $2\alpha = n - 2w$ and $2\beta + 1 = \sqrt{(n - 2w + 1)^2 + (w - 1)}$.

Lemma 41. If 1-perfect code exists in $J(2w + a, w)$ then

$$\Phi_1(w, a) = (\beta^2 - \alpha^2 + 1)((\beta + 1)^2 - \alpha^2 + 1),$$

- $\gcd((\beta^2 - \alpha^2 + 1)((\beta + 1)^2 - \alpha^2 + 1)) = 1$
- $\beta^2 - \alpha^2 + 1$ is squarefree
- $(\beta + 1)^2 - \alpha^2 + 1$ is squarefree

where $2\alpha = n - 2w$ and $2\beta + 1 = \sqrt{(n - 2w + 1)^2 + (w - 1)}$.

Proof. In the proof of the Theorem 38 it was shown that if 1-perfect code exists in $J(2w + a, w)$, and its strength is $w - d$ for some integer d , then

$$\Phi_1(w, a) = (w - d + 1)(w + a + d).$$

Since $d = w - (\beta - \alpha)(\beta + \alpha) = (\beta - \alpha)(\beta + \alpha + 1) + 1 - (\beta - \alpha)(\beta + \alpha) = \beta - \alpha + 1$,

$$w - d + 1 = (\beta - \alpha)(\beta + \alpha) + 1 = \beta^2 - \alpha^2 + 1,$$

$$w + a + d = (\beta - \alpha)(\beta + \alpha + 1) + 1 + 2\alpha + \beta - \alpha + 1 = (\beta + 1)^2 - \alpha^2 + 1,$$

the expression for $\Phi_1(w, a)$ is

$$\Phi_1(w, a) = (w - d + 1)(w + a + d).$$

Gordon [20] proved, that $\Phi_1(w, a)$ must be squarefree, which proves the lemma.

□

2.2 Moments

2.2.1 Introduction

2.2.1.1 Configuration distribution

The following definitions appear in [18].

Let C be a code in $J(n, w)$. We can partition the coordinate set N into r subsets $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$. A vector $x \in V_w^n$ can be written as $x = (x_1, x_2, \dots, x_r)$, where $x_i \in \alpha_i$, $1 \leq i \leq r$. We say that x is from *configuration* (w_1, w_2, \dots, w_r) , $\sum_{i=1}^r w_i = w$, if $|x_i| = w_i$, $1 \leq i \leq r$. We denote by $D_{(w_1, w_2, \dots, w_r)}$ the number of codewords from configuration (w_1, w_2, \dots, w_r) . The *configuration distribution of C* is a vector consisting of all the values $D_{(w_1, w_2, \dots, w_r)}$, where $w_i \leq |\alpha_i|$, $1 \leq i \leq r$, and $\sum_{i=1}^r w_i = w$.

In [15] several partitions with $r = 2$ were considered. The most important one is the one in which $|\alpha_1| = w$ and $|\alpha_2| = w + a$. Clearly, permutation on the columns of e -perfect code C will result in an e -perfect code isomorphic to C . In this case it was proved in [15] that an e -perfect code have exactly $e + 1$ different configuration distributions.

In order to avoid confusion we will assume that the vector from configuration $(w, 0)$ is always a codeword in a perfect code C . If we permute the columns of C (in other words, we take another partition $\{\beta_1, \beta_2\}$ of N , such that $|\beta_1| = w$ and $|\beta_2| = w + a$) in a way that the vector from configuration $(w, 0)$ is not a codeword we will call the obtained code a *translate* of C . For each j , $1 \leq j \leq e$, there exists a translate with exactly one *translate-word* from configuration $(w - j, j)$, and no translate -word from configuration $(w - i, i)$, $0 \leq i \leq e$, $i \neq j$. The translate -word from configuration $(w - j, j)$ will be called a *translate leader*.

Let A_i , $0 \leq i \leq w$, be the number of codewords in configuration $(w - i, i)$ and let $B_{i,j}$, $0 \leq i \leq w$, $0 \leq j \leq e$, be the number of translate-words from configuration $(w - i, i)$ in the translate with translate-leader $(w - j, j)$. Note, that $A_i = B_{i,0}$ and $B_{i,j} = D_{(w-i,i)}$ in the corresponding translate. A_i is also the number of codewords which have distance i to the codeword from configuration $(w, 0)$ and $(A_i)_{i=0}^w$ is the inner distance distribution of the code in the Johnson scheme. $(B_i)_{i=0}^w$ is the configuration distribution which is akin to the weight distribution in the Hamming scheme.

Etzion in [18] proved the following theorem:

Theorem 42. For a given e -perfect code C in $J(n, w)$ we have

$$\sum_{j=0}^e \binom{w}{j} \binom{w+a}{j} B_{i,j} = \binom{w}{i} \binom{w+a}{i}.$$

2.2.1.2 Moments.

In [18] Etzion defined a generalization for *moments* of a code which was given for the Hamming scheme [21].

Let C be an e -perfect code in $J(n, w)$, and let $\{\alpha_1, \alpha_2\}$ be a partition of N such that $|\alpha_1| = k$ and $|\alpha_2| = n - k$. Let A_i be the number of codewords from configuration $(i, w - i)$ (note, that this definition is slightly different from the one in the previous definition). Let $\{\beta_1, \beta_2\}$ be another partition of N such that $|\beta_1| = k$ and $|\beta_2| = n - k$, and B_i be the number of codewords from configuration $(i, w - i)$ with respect to this partition.

The r -th *power moment*, $0 \leq r$, of C with respect to these partitions is defined by

$$\sum_{i=0}^k i^r A_i, \sum_{i=0}^k i^r B_i$$

and the r -th *binomial moment*, $0 \leq r$, of C is defined by

$$\sum_{i=0}^k \binom{i}{r} A_i, \sum_{i=0}^k \binom{i}{r} B_i.$$

We define the *difference configuration distributions* between the two partitions by $\Delta_i = A_i - B_i$, $0 \leq i \leq k$. The r -th power moments and the r -th binomial moments with respect to the difference configuration distributions are defined by

$$\sum_{i=0}^k i^r \Delta_i, \sum_{i=0}^k \binom{i}{r} \Delta_i.$$

Two types of moments are connected by *Stirling number of the second kind* $S(r, v)$. $S(r, v)$, $r \geq v \geq 0$ is the number of ways to partition a set of r elements into v nonempty sets. The following are known three formulas [22] :

$$S(r, v) = \frac{1}{v!} \sum_{i=0}^r (-1)^{v-i} \binom{v}{i} i^r,$$

$$S(r, v) = S(r-1, v-1) + vS(r-1, v),$$

where $S(r, 1) = S(r, r) = 1$ and $S(r, 0) = 0$ for $r > 0$,

$$i^r = \sum_{v=0}^r v! \binom{i}{v} S(r, v).$$

Hence

$$\sum_{i=0}^k i^r \Delta_i = \sum_{i=0}^k \sum_{v=0}^r v! \binom{i}{v} S(r, v) \Delta_i = \sum_{v=0}^r v! S(r, v) \sum_{i=0}^k \binom{i}{v} \Delta_i.$$

Therefore, it can be proved by induction that

Theorem 43. For a given integer t , $\sum_{i=0}^k i^r \Delta_i = 0$ for all $0 \leq r \leq t$ if and only if $\sum_{i=0}^k \binom{i}{r} \Delta_i = 0$ for all $0 \leq r \leq t$.

In [18] Etzion showed that for $r \leq \varphi$, where φ is a strength of the code, the values of the binomial moments can be easily computed.

Lemma 44. If C is a perfect code in $J(n, w)$ and φ is its strength, then for each r , $0 \leq r \leq \varphi$ we have

$$\sum_{i=0}^k \binom{i}{r} A_i = \sum_{i=0}^k \binom{i}{r} B_i = \binom{k}{r} \frac{\binom{n-r}{w-r}}{\Phi_e(n, w)}.$$

Corollary 45. If C is a perfect code in $J(n, w)$ and φ is its strength, then for each r , $0 \leq r \leq \varphi$ we have $\sum_{i=0}^k \binom{i}{r} \Delta_i = 0$ and $\sum_{i=0}^k i^r \Delta_i = 0$.

2.2.2 Binomial moments for 1-perfect codes in $J(n, w)$

We saw in the previous section that for $r \leq \varphi$, where φ is a strength of the code, the values of the binomial moments can be easily computed. In this section we consider the binomial moments for $r > \varphi$, for 1-perfect codes in $J(n, w)$.

In [18] Etzion proved the following lemma.

Lemma 46. Given $\{H_1, H_2\}$ partition of N such that $|H_1| = k$, $|H_2| = n - k$, for any i , $0 \leq i \leq k$ we have

$$(i+1)(w+a-k+i+1)A_{i+1} + [(1+i(k-1)) + (w-i)(w+a-k+i)]A_i$$

$$+ (k-i+1)(w-i+1)A_{i-1} = \binom{k}{i} \binom{2w+a-k}{w-i},$$

where A_i is the number of codewords from configuration $(i, w-i)$.

Let $\{\alpha_1, \alpha_2\}$ be a partition of N such that $|\alpha_1| = w$, $|\alpha_2| = n - w$, and a vector of $(w, 0)$ configuration be a codeword. Let A_i be the number of codewords from configuration $(i, w-i)$. Let $\{\beta_1, \beta_2\}$ be another partition of N such that $|\beta_1| = w$, $|\beta_2| = n - w$, let B_i be the number of codewords from configuration $(i, w-i)$ with respect to this partition, and let $\Delta_i = A_i - B_i$, $0 \leq i \leq w$.

Theorem 47. If C is a 1-perfect code in $J(n, w)$ and φ is its strength, then for each k , $\varphi < k \leq w$, we have

$$\begin{aligned}\sum_{i=0}^w \binom{i}{k} \Delta_i &= (-1)^{w-k} \prod_{l=1}^{w-k} \frac{[(l-1)n + l^2 - l + 1 - w(2l-1)]}{l^2} \\ \sum_{i=0}^w \binom{i}{k} B_i &= \frac{\binom{n-w}{k} \binom{n-k}{w-k} - (-1)^{w-k} \prod_{l=1}^{w-k} \frac{[(l-1)n + l^2 - l + 1 - w(2l-1)]}{l^2}}{\Phi_1(n, w)} \\ \sum_{i=0}^w \binom{i}{k} A_i &= \frac{w(n-w)(-1)^{w-k} \prod_{l=1}^{w-k} \frac{[(l-1)n + l^2 - l + 1 - w(2l-1)]}{l^2} + \binom{n-w}{k} \binom{n-k}{w-k}}{\Phi_1(n, w)}.\end{aligned}$$

Proof. Assume that C is a 1-perfect code in $J(n, w)$ and φ is its strength. By Lemma 46 we have

$$\begin{aligned}\binom{w}{i} \binom{n-w}{w-i} &= A_{i+1}(i+1)(i+1+n-2w) \\ &\quad + A_i(1+(w-i)(n-2w+2i)) + A_{i-1}(w-i+1)^2 \\ \binom{w}{i} \binom{n-w}{w-i} &= B_{i+1}(i+1)(i+1+n-2w) \\ &\quad + B_i(1+(w-i)(n-2w+2i)) + B_{i-1}(w-i+1)^2\end{aligned}$$

where $0 \leq i \leq w$.

Therefore,

$$\begin{aligned}0 &= \Delta_{i+1}(i+1)(i+1+n-2w) + \Delta_i(1+(w-i)(n-2w+2i)) \\ &\quad + \Delta_{i-1}(w-i+1)^2,\end{aligned}$$

or

$$\begin{aligned}0 &= \Delta_{i+1}[(i+1)^2 + (n-2w)(i+1)] + \Delta_i[1 + w(n-2w) + i(4w-n) - 2i^2] \\ &\quad + \Delta_{i-1}[w^2 - 2w(i-1) + (i-1)^2]\end{aligned}$$

Multiply it by $\binom{i}{k}$ and sum over all i , $0 \leq i \leq w$:

$$\begin{aligned}
0 &= \sum_{i=0}^w \binom{i}{k} (i+1)^2 \Delta_{i+1} + (n-2w) \sum_{i=0}^w \binom{i}{k} (i+1) \Delta_{i+1} + \sum_{i=0}^w \binom{i}{k} \Delta_i \\
&+ w(n-2w) \sum_{i=0}^w \binom{i}{k} \Delta_i + (4w-n) \sum_{i=0}^w \binom{i}{k} i \Delta_i \\
&- 2 \sum_{i=0}^w \binom{i}{k} i^2 \Delta_i + w^2 \sum_{i=0}^w \binom{i}{k} \Delta_{i-1} \\
&- 2w \sum_{i=0}^w \binom{i}{k} (i-1) \Delta_{i-1} + \sum_{i=0}^w \binom{i}{k} (i-1)^2 \Delta_{i-1}
\end{aligned}$$

We prove the following proposition (see Appendix A).

Proposition 48. For each k , $\varphi < k \leq w$, we have

$$0 = [1 + k^2 - k(1+n) + nw - w^2] \sum_{i=0}^w \binom{i}{k} \Delta_i + (1-k+w)^2 \sum_{i=0}^w \binom{i}{k-1} \Delta_i. \quad (2.5)$$

Note that $\sum_{i=0}^w \binom{i}{w} \Delta_i = \Delta_w = 1$.

If we assume that $k = w$, from (2.5) we get:

$$0 = (1-w) \sum_{i=0}^w \binom{i}{w} \Delta_i + \sum_{i=0}^w \binom{i}{w-1} \Delta_i$$

therefore,

$$\sum_{i=0}^w \binom{i}{w-1} \Delta_i = -(1-w)$$

If we assume that $k = w-1$, from (2.5) we get:

$$0 = (3+n-3w) \sum_{i=0}^w \binom{i}{w-1} \Delta_i + 2^2 \sum_{i=0}^w \binom{i}{w-2} \Delta_i$$

therefore,

$$\sum_{i=0}^w \binom{i}{w-2} \Delta_i = \frac{(n+3-3w)(1-w)}{2^2}$$

In general, for $k = w - j$ from (2.5) we get:

$$0 = [jn + (j^2 + j + 1) - w(2j + 1)] \sum_{i=0}^w \binom{i}{w-j} \Delta_i + (j+1)^2 \sum_{i=0}^w \binom{i}{w-j-1} \Delta_i$$

Therefore,

$$\begin{aligned} \sum_{i=0}^w \binom{i}{w-j-1} \Delta_i &= - \frac{[jn + (j^2 + j + 1) - w(2j + 1)] \sum_{i=0}^w \binom{i}{w-j} \Delta_i}{(j+1)^2} \\ \sum_{i=0}^w \binom{i}{w-j} \Delta_i &= (-1)^j \prod_{l=1}^j \frac{[(l-1)n + l^2 - l + 1 - w(2l-1)]}{l^2} \end{aligned}$$

or

$$\sum_{i=0}^w \binom{i}{k} \Delta_i = (-1)^{w-k} \prod_{l=1}^{w-k} \frac{[(l-1)n + l^2 - l + 1 - w(2l-1)]}{l^2}$$

for $k = \varphi + 1, \dots, w$, where φ is the strength of the code.

Since $\Delta_i = A_i - B_i$ and by Theorem 42 $\binom{w}{i} \binom{n-w}{i} = A_i + w(n-w)B_i$, we have:

$$\begin{aligned} \sum_{i=0}^w \binom{i}{k} \Delta_i &= \sum_{i=0}^w \binom{i}{k} A_i - \sum_{i=0}^w \binom{i}{k} B_i \\ &= \sum_{i=0}^w \binom{i}{k} \binom{w}{i} \binom{n-w}{i} - w(n-w) \sum_{i=0}^w \binom{i}{k} B_i - \sum_{i=0}^w \binom{i}{k} B_i \end{aligned}$$

therefore, since $\sum_{i=0}^w \binom{i}{k} \binom{w}{i} \binom{n-w}{i} = \binom{n-w}{k} \binom{n-k}{w-k}$,

$$\begin{aligned} \sum_{i=0}^w \binom{i}{k} B_i &= \frac{\sum_{i=0}^w \binom{i}{k} \binom{w}{i} \binom{n-w}{i} - \sum_{i=0}^w \binom{i}{k} \Delta_i}{w(n-w) + 1} = \frac{\binom{n-w}{k} \binom{n-k}{w-k} - \sum_{i=0}^w \binom{i}{k} \Delta_i}{w(n-w) + 1} \\ &= \frac{\binom{n-w}{k} \binom{n-k}{w-k} - (-1)^{w-k} \prod_{l=1}^{w-k} \frac{[(l-1)n + l^2 - l + 1 - w(2l-1)]}{l^2}}{w(n-w) + 1} \end{aligned}$$

$$\sum_{i=0}^w \binom{i}{k} A_i = \frac{w(n-w) \left((-1)^{w-k} \prod_{l=1}^{w-k} \frac{[(l-1)n + l^2 - l + 1 - w(2l-1)]}{l^2} \right) + \binom{n-w}{k} \binom{n-k}{w-k}}{w(n-w) + 1}$$

where $k = \varphi + 1, \dots, w$, and φ is the strength of the code. □

Note that if in expression (2.5) we assume that $k = \varphi + 1$, then the second summand disappears, and the coefficient of the first summand must be 0. Therefore, we got equation for φ , and its solution gives us the expression for the strength of a 1-perfect code:

$$\varphi = \frac{n - 1 - \sqrt{(n - 2w + 1)^2 + 4(w - 1)}}{2}.$$

Therefore, binomial moments is a second way to get the strength of the perfect code.

2.2.2.1 Applications of Binomial moments for 1-perfect codes in $J(n, w)$

Now we consider the $(w - 5)$ -binomial moment and several partitions of set of coordinates N in order to exclude a number of parameters for 1-perfect code.

We examine the $(w - 5)$ -binomial moment with respect to the difference configuration distributions:

$$\sum \binom{i}{w-5} \Delta_i = \frac{(w-1)(a-w+3)(2a-w+7)(3a-w+13)(4a-w+21)}{(5!)^2}$$

Note that binomial moments must be integer number, therefore we have one of divisibility conditions for 1-perfect code.

In addition we examine the following three partitions of set of coordinates:

1. $\{\alpha_1, \alpha_2\}$, such that $|\alpha_1| = w$, $|\alpha_2| = n - w$, and the vector of $(w, 0)$ configuration is a codeword. Let A_i be the number of codewords from configuration $(i, w - i)$ with respect to this partition. By Lemma 46 and using the fact that $A_w = 1$, $A_{w-1} = 0$ we obtain the following expression

$$\begin{aligned} A_{w-5} &= \frac{w(w-1)(w+a)(w+a-1)}{(5!)^2} [a^2(26 + (w-9)w) \\ &+ (w-3)(-181 + w(87 + (w-15)w)) \\ &+ a(-221 + w(132 + w(2w-27)))] \end{aligned}$$

2. $\{\beta_1, \beta_2\}$, such that $|\beta_1| = w - 2$, $|\beta_2| = n - w + 2$, and the vector of $(w - 2, 2)$ configuration is a codeword. Let B_i be the number of codewords from configuration

$(i, w-i)$ with respect to this partition. By Lemma 46 and using the fact that $B_{w-2} = 1$, $B_{w-3} = \frac{(w+a)(w+a-1)}{6}$ we obtain the following expression

$$B_{w-5} = \frac{1}{15 * 48} (w+a-1)(w+a)[a^2(26+(w-9)w) + (w-3)(19+w(-3+(w-5)w)) + a(-21+w(42+w(2w-17)))]$$

3. $\{\gamma_1, \gamma_2\}$, such that $|\gamma_1| = w+2$, $|\gamma_2| = n-w-2$, and the vector of $(w, 0)$ configuration is a codeword. Let C_i be the number of codewords from configuration $(i, w-i)$ with respect to this partition. By Lemma 46 and using the fact that $C_w = 1$, $C_{w-1} = \frac{w(w-1)}{6}$ we obtain the following expression

$$C_{w-3} = \frac{1}{15 * 48} w(w-1)[a^2(-4+(w+1)w) + (w-3)(19+w(-3+(w-5)w + a(49+w(-18+w(2w-7)))))]$$

We chose those expressions since one of the factors of all the denominators is '5'.

Since we know that $w \equiv w+a \equiv 1 \pmod{12}$ or $w \equiv w+a \equiv 7 \pmod{12}$ we consider all possible cases for w and $w+a$ modulo 60.

Using the above four divisibility conditions, we build two tables w versus $w+a$ modulo 60, where $w \equiv w+a \equiv 1 \pmod{12}$ and $w \equiv w+a \equiv 7 \pmod{12}$, respectively, where '-' denotes that there are no 1-perfect codes with such parameters.

Table 2.1: $w \equiv w+a \equiv 1 \pmod{12}$

$w+a$ w	1	13	25	37	49
1			—		—
13			—	—	—
25		—	—		
37		—		—	—
49	—	—		—	—

Table 2.2: $w \equiv w + a \equiv 7(\text{mod } 12)$

$\begin{matrix} w+a \\ w \end{matrix}$	7	19	31	43	55
7	—	—		—	
19	—	—	—	—	
31		—			
43	—	—			—
55				—	—

In addition, if we write $w = 60k + i$ and $w + a = 60y + j$ for $i, j \in \{1, 13, 25, 37, 49, 7, 19, 31, 43, 55\}$ then we get the following existence conditions:

- If there exists 1-perfect code with $w \equiv w + a \equiv 13(\text{mod } 60)$ then $k + y \equiv 3(\text{mod } 5)$.
- If there exists 1-perfect code with $w \equiv 25(\text{mod } 60)$ and $w + a \equiv 1(\text{mod } 60)$ then $y \equiv 0(\text{mod } 5)$.
- If there exists 1-perfect code with $w \equiv 25(\text{mod } 60)$ and $w + a \equiv 37(\text{mod } 60)$ then $2k - y \equiv 2(\text{mod } 5)$.
- If there exists 1-perfect code with $w \equiv 37(\text{mod } 60)$ and $w + a \equiv 25(\text{mod } 60)$ then $4k - 3y \equiv 4(\text{mod } 5)$.
- If there exists 1-perfect code with $w \equiv 7(\text{mod } 60)$ and $w + a \equiv 55(\text{mod } 60)$ then $4k - 3y \equiv 0(\text{mod } 5)$ and $a \equiv 0(\text{mod } 24)$.
- If there exists 1-perfect code with $w \equiv 31(\text{mod } 60)$ and $w + a \equiv 55(\text{mod } 60)$ then $k \equiv 2(\text{mod } 5)$ and $a \equiv 0(\text{mod } 24)$.
- If there exists 1-perfect code with $w \equiv 43(\text{mod } 60)$ and $w + a \equiv 43(\text{mod } 60)$ then $k + y \equiv 2(\text{mod } 5)$ and $a \equiv 0(\text{mod } 24)$.
- If there exists 1-perfect code with $w \equiv 55(\text{mod } 60)$ and $w + a \equiv 7(\text{mod } 60)$ then $2k - y \equiv 0(\text{mod } 5)$ and $a \equiv 0(\text{mod } 24)$.
- If there exists 1-perfect code with $w \equiv 55(\text{mod } 60)$ and $w + a \equiv 31(\text{mod } 60)$ then $y \equiv 2(\text{mod } 5)$ and $a \equiv 0(\text{mod } 24)$.

2.2.3 Binomial moments for 2-perfect code in $J(2w, w)$

In this section we calculate the expression for k -th binomial moments with respect to the difference configuration distributions for all $k > \varphi$, where φ is a strength of a 2-perfect code in $J(2w, w)$, and obtain expression for strength of this code.

Let C be a 2-perfect code in $J(2w, w)$. Let $\{\alpha_1, \alpha_2\}$ be a partition of N such that $|\alpha_1| = w$, $|\alpha_2| = w$, and vector of $(w, 0)$ configuration is a codeword. Let A_i be the number of codewords from configuration $(i, w-i)$. Let $\{\beta_1, \beta_2\}$ be another partition of N such that $|\beta_1| = w$, $|\beta_2| = w$, and let B_i be number of codewords from configuration $(i, w-i)$ with respect to this partition. The k -th binomial moment, $0 \leq k$, of C is defined by

$$\sum_{i=0}^k \binom{i}{k} A_i, \quad \sum_{i=0}^k \binom{i}{k} B_i.$$

By considering how $\binom{w}{i} \binom{w}{w-i}$ vectors from configuration $(i, w-i)$ are 2-covered by C we obtain the following formulas for any i , $0 \leq i \leq w$:

$$\begin{aligned} \binom{w}{i}^2 &= \binom{i+2}{2}^2 A_{i+2} + \binom{w-i+2}{2}^2 A_{i-2} \\ &+ \left[(i+1)^2 + 2(i+1)(w-i-1) \binom{i+1}{2} \right] A_{i+1} \\ &+ \left[(w-i+1)^2 + 2(i-1)(w-i+1) \binom{w-1+1}{2} \right] A_{i-1} \\ &+ \left[1 + 2i(w-i) + 2 \binom{i}{2} \binom{w-i}{2} + i^2(w-i)^2 \right] A_i \end{aligned}$$

$$\begin{aligned} \binom{w}{i}^2 &= \binom{i+2}{2}^2 B_{i+2} + \binom{w-i+2}{2}^2 B_{i-2} \\ &+ \left[(i+1)^2 + 2(i+1)(w-i-1) \binom{i+1}{2} \right] B_{i+1} \\ &+ \left[(w-i+1)^2 + 2(i-1)(w-i+1) \binom{w-1+1}{2} \right] B_{i-1} \\ &+ \left[1 + 2i(w-i) + 2 \binom{i}{2} \binom{w-i}{2} + i^2(w-i)^2 \right] B_i \end{aligned}$$

Let $\Delta_i = A_i - B_i$, for $0 \leq i \leq w$. Hence we obtain:

$$\begin{aligned}
0 &= \binom{i+2}{2}^2 \Delta_{i+2} + \binom{w-i+2}{2}^2 \Delta_{i-2} \\
&+ \left[(i+1)^2 + 2(i+1)(w-i-1) \binom{i+1}{2} \right] \Delta_{i+1} \\
&+ \left[(w-i+1)^2 + 2(i-1)(w-i+1) \binom{w-i+1}{2} \right] \Delta_{i-1} \\
&+ \left[1 + 2i(w-i) + 2 \binom{i}{2} \binom{w-i}{2} + i^2(w-i)^2 \right] \Delta_i
\end{aligned}$$

Next we multiply it by $\binom{i}{k}$ and sum over all $0 \leq i \leq w$:

$$\begin{aligned}
0 &= \sum_{i=0}^w \binom{i}{k} \binom{i+2}{2}^2 \Delta_{i+2} + \sum_{i=0}^w \binom{i}{k} \binom{w-i+2}{2}^2 \Delta_{i-2} \\
&+ \sum_{i=0}^w \binom{i}{k} \left[(i+1)^2 + 2(i+1)(w-i-1) \binom{i+1}{2} \right] \Delta_{i+1} \\
&+ \sum_{i=0}^w \binom{i}{k} \left[(w-i+1)^2 + 2(i-1)(w-i+1) \binom{w-i+1}{2} \right] \Delta_{i-1} \\
&+ \sum_{i=0}^w \binom{i}{k} \left[1 + 2i(w-i) + 2 \binom{i}{2} \binom{w-i}{2} + i^2(w-i)^2 \right] \Delta_i
\end{aligned}$$

We prove the following proposition (see Appendix B).

Proposition 49. For each k , $\varphi < k \leq w$, we have

$$\begin{aligned}
0 &= \frac{1}{4}(4 + k^4 + 5w^2 - 2w^3 + w^4 - 2k^3(1 + 2w) + k^2(7 + 2w + 6w^2)) \\
&- 2k(3 + 5w - w^2 + 2w^3) \sum_{i=0}^w \binom{i}{k} \Delta_i \\
&+ \frac{1}{2}(1 - k + w)^2(4 + k^2 + w^2 - 2k(1 + w)) \sum_{i=0}^w \binom{i}{k-1} \Delta_i \\
&+ \frac{1}{4}(1 - k + w)^2(2 - k + w)^2 \sum_{i=0}^w \binom{i}{k-2} \Delta_i
\end{aligned}$$

Note, that from this formula we can derive the expression for strength of 2-perfect code in $J(2w, w)$ by substitution $k = \varphi + 1$. Hence, we assume that $\sum \binom{i}{j} \Delta_i = 0$ for all $j < k$, and $\sum \binom{i}{k} \Delta_i \neq 0$. Thus we obtain the following four roots:

$$k = \frac{1}{2} \left(1 + 2w \mp \sqrt{-11 + 8w \mp 4\sqrt{5 - 6w + 2w^2}} \right)$$

or

$$\varphi = \frac{1}{2} \left(-1 + 2w \mp \sqrt{-11 + 8w \mp 4\sqrt{5 - 6w + 2w^2}} \right).$$

If we assume that $k = w - j + 2$ we obtain the following recursion formula:

$$\sum_{i=0}^w \binom{i}{w-j} \Delta_i = - \frac{F(w, j) \sum_{i=0}^w \binom{i}{w-j+2} \Delta_i + G(w, j) \sum_{i=0}^w \binom{i}{w-j+1} \Delta_i}{(j-1)^2 j^2}$$

where $2 \leq j < w - \varphi$, $F(w, j) = 20 + (j-3)j(10 + (j-3)j) - 14w - 4(j-3)jw + 2w^2$ and $G(w, j) = 2(j-1)^2(4 + (j-2)j - 2w)$.

Since we consider the case $e = 2$, we have two possibilities for B_i : $B_{i,1}$ and $B_{i,2}$. In other words, we consider the number of translate-words from configuration $(i, w-i)$ in the translate with translate-leader $(w-1, 1)$ and $(w-2, 2)$, respectively.

Thus we have two possible Δ_i : $\Delta_{i,1}$ and $\Delta_{i,2}$.

Now we compute binomial moments for the first several values of j .

From $\Delta_{w,l} = 1$, for $l = 1, 2$, we have $\sum_{i=0}^w \binom{i}{w} \Delta_{i,l} = \Delta_{w,l} = 1$.

From $\Delta_{w-1,1} = -1$, $\Delta_{w-1,2} = 0$, it follows:

$$\sum_{i=0}^w \binom{i}{w-1} \Delta_{i,l} = \binom{w-1}{w-1} \Delta_{w-1,l} + \binom{w}{w-1} \Delta_{w,l} = \begin{cases} w-1, & l=1 \\ w, & l=2 \end{cases}$$

- $j = 2$.

$$\begin{aligned} \sum_{i=0}^w \binom{i}{w-2} \Delta_{i,1} &= \frac{(w-1)(w-2)}{2} \\ \sum_{i=0}^w \binom{i}{w-2} \Delta_{i,2} &= \frac{(w+1)(w-2)}{2} \end{aligned}$$

- $j = 3$.

$$\begin{aligned} \sum_{i=0}^w \binom{i}{w-3} \Delta_{i,1} &= \frac{(w-1)(w-2)(w-3)}{6} \\ \sum_{i=0}^w \binom{i}{w-3} \Delta_{i,2} &= \frac{(w-2)(3w^2 - 5w - 14)}{2 * 3^2} \end{aligned}$$

- $j = 4$.

$$\sum_{i=0}^w \binom{i}{w-4} \Delta_{i,1} = \frac{(w-1)(w-2)(w-5)(5w-14)}{3^2 4^2}$$

$$\sum_{i=0}^w \binom{i}{w-4} \Delta_{i,2} = \frac{(w-2)(w-5)(5w^2-7w-26)}{3^2 4^2}$$

- $j = 5$.

$$\sum_{i=0}^w \binom{i}{w-5} \Delta_{i,1} = \frac{(w-1)(w-2)(w-5)(334-171w+7w^2)}{3^2 4^2 5^2}$$

$$\sum_{i=0}^w \binom{i}{w-5} \Delta_{i,2} = \frac{(w-2)(w-5)(17w^3-147w^2+66w+680)}{3^2 4^2 5^2}.$$

Note that by [14], if 2-perfect code exists in $J(2w, w)$, then $w \equiv 2, 26$ or $50 \pmod{60}$. But for $w \equiv 26 \pmod{60}$ the last divisibility condition is not satisfied, therefore remains only $w \equiv 2$ or $50 \pmod{60}$.

- $j = 6$.

$$\sum_{i=0}^w \binom{i}{w-6} \Delta_{i,1} = \frac{2(w-1)(w-2)(w-5)(-5684+3544w-589w^2+29w^3)}{3^2 4^2 5^2 6^2}$$

$$\sum_{i=0}^w \binom{i}{w-6} \Delta_{i,2} = \frac{2(w-2)(w-5)(-12228+228w+2663w^2-548w^3+29w^4)}{3^2 4^2 5^2 6^2}.$$

- $j = 7$.

$$\sum_{i=0}^w \binom{i}{w-7} \Delta_{i,1} = \frac{2(w-1)(w-2)(w-5)}{3^2 4^2 5^2 6^2 7^2}$$

$$* (262324 - 185444w + 39797w^2 - 3376w^3 + 99w^4)$$

$$\sum_{i=0}^w \binom{i}{w-7} \Delta_{i,2} = \frac{2(w-2)(w-5)}{3^2 4^2 5^2 6^2 7^2}$$

$$* (585224 - 59628w - 123650w^2 + 34855w^3 - 3236w^4 + 99w^5).$$

The last divisibility conditions leave only the following values of w modulo 420:

- $w \equiv 2, 302$ or $362 \pmod{420}$;
- $w \equiv 50, 110$ or $170 \pmod{420}$.

2.2.3.1 Necessary conditions for the existence of a 2-perfect code in $J(2w, w)$

In this section we show the necessary conditions for the existence of a 2-perfect code in $J(2w, w)$ using Pell equation and prove that there are no 2-perfect codes in $J(2w, w)$ for $n < 2.5 * 10^{15}$.

Assume C is a 2-perfect code in $J(2w, w)$.

We saw that the strength of the code is:

$$\frac{1}{2}(-1 + 2w - \sqrt{8w - 11 \pm 4\sqrt{5 - 6w + 2w^2}}).$$

Hence, the first constraint is:

$$\sqrt{5 - 6w + 2w^2} \in \mathbb{Z}$$

therefore, $\exists y \in \mathbb{Z}$, s.t.

$$\begin{aligned} 5 - 6w + 2w^2 &= y^2 \\ 10 - 12w + 4w^2 &= 2y^2 \\ (2w - 3)^2 - 2y^2 &= -1 \end{aligned}$$

Let $x = 2w - 3$. This brings us to the Pell equation:

$$x^2 - 2y^2 = -1$$

with the family of solutions in the form of:

$$x = \frac{(1 + \sqrt{2})^k + (1 - \sqrt{2})^k}{2} \quad (2.6)$$

$$y = \frac{(1 + \sqrt{2})^k - (1 - \sqrt{2})^k}{2\sqrt{2}} \quad (2.7)$$

where k is odd [23].

Using the binomial formula, from (2.6) and denoting $k = 2m + 1$ we derive the following expression for x :

$$\begin{aligned} x &= \frac{1}{2} \left[\sum_{i=0}^{2m+1} \binom{2m+1}{i} 2^{\frac{i}{2}} + \sum_{i=0}^{2m+1} \binom{2m+1}{i} 2^{\frac{i}{2}} (-1)^i \right] \\ &= \sum_{i \text{ is even}} \binom{2m+1}{i} 2^{\frac{i}{2}} = \sum_{j=0}^m \binom{2m+1}{2j} 2^j \end{aligned}$$

or

$$x = 1 + \binom{2m+1}{2}2 + \binom{2m+1}{4}2^2 + \dots + \binom{2m+1}{2m}2^m.$$

We know from [14] that if a 2-perfect code exists in $J(2w, w)$, then $w \equiv 2, 26, 50 \pmod{60}$, and thus $w \equiv 2 \pmod{12}$.

Since $w = \frac{x+3}{2}$, then $\exists z$, s.t. $12z = w - 2 = \frac{x+3}{2} - 2 = \frac{x-1}{2}$. Consequently $24z = x - 1$, $x \equiv 1 \pmod{24}$, and in particular, $x \equiv 1 \pmod{4}$ and $x \equiv 1 \pmod{3}$.

- Since $x \equiv 1 \pmod{4}$ we have:

$$1 + \binom{2m+1}{2}2 \equiv 1 \pmod{4}$$

or

$$2m(2m+1) \equiv 0 \pmod{4}$$

therefore m is even. Denote $m = 2t$.

- Since $2 \equiv -1 \pmod{3}$, we have:

$$2^j \equiv \begin{cases} 2, & j \text{ is odd} \\ 1, & j \text{ is even} \end{cases} \pmod{3}$$

therefore, from $x \equiv 1 \pmod{3}$:

$$\sum_{j \text{ is even}} \binom{2m+1}{2j} + 2 \sum_{j \text{ is odd}} \binom{2m+1}{2j} \equiv 0 \pmod{3}$$

or

$$\sum_{j \text{ is even}} \binom{2m+1}{2j} - \sum_{j \text{ is odd}} \binom{2m+1}{2j} \equiv 0 \pmod{3}$$

For example, for $m = 6$, we obtain the contradiction:

$$\left[\binom{13}{2} + \binom{13}{6} + \binom{13}{10} \right] - \left[\binom{13}{4} + \binom{13}{8} + \binom{13}{12} \right] = 65 \not\equiv 0 \pmod{3}.$$

The second constraint is:

$$\sqrt{8w - 11 \pm 4\sqrt{5 - 6w + 2w^2}} \in \mathbb{Z}.$$

We examine two cases, positive root and negative root.

- $\sqrt{8w - 11 + 4\sqrt{5 - 6w + 2w^2}} \in \mathbb{Z}.$

$$\begin{aligned}
8w - 11 + 4\sqrt{5 - 6w + 2w^2} &= 8w - 11 + 4y \\
&= 8\left(\frac{x+3}{2}\right) - 11 + 4y \\
&= 4x + 1 + 4y = 4(x+y) + 1
\end{aligned}$$

therefore, $\exists c \in \mathbb{Z}$, s.t.

$$4(x+y) + 1 = c^2$$

- $\sqrt{8w - 11 - 4\sqrt{5 - 6w + 2w^2}} \in \mathbb{Z}.$

$$\begin{aligned}
8w - 11 - 4\sqrt{5 - 6w + 2w^2} &= 8w - 11 - 4y \\
&= 8\left(\frac{x+3}{2}\right) - 11 - 4y \\
&= 4x + 1 - 4y = 4(x-y) + 1
\end{aligned}$$

therefore, $\exists d \in \mathbb{Z}$, s.t.

$$4(x-y) + 1 = d^2$$

From (2.6) and (2.7) we obtain:

$$\begin{aligned}
x+y &= \frac{\sqrt{2}(1+\sqrt{2})^k + \sqrt{2}(1-\sqrt{2})^k + (1+\sqrt{2})^k - (1-\sqrt{2})^k}{2\sqrt{2}} \\
&= \frac{(\sqrt{2}+1)(1+\sqrt{2})^k + (\sqrt{2}-1)(1-\sqrt{2})^k}{2\sqrt{2}} \\
&= \frac{(1+\sqrt{2})^{k+1} - (1-\sqrt{2})^{k+1}}{2\sqrt{2}} \tag{2.8}
\end{aligned}$$

$$\begin{aligned}
x-y &= \frac{\sqrt{2}(1+\sqrt{2})^k + \sqrt{2}(1-\sqrt{2})^k - (1+\sqrt{2})^k + (1-\sqrt{2})^k}{2\sqrt{2}} \\
&= \frac{(\sqrt{2}-1)(\sqrt{2}+1)^k - (\sqrt{2}+1)(\sqrt{2}-1)^k}{2\sqrt{2}} \\
&= \frac{(\sqrt{2}+1)^{k-1} - (\sqrt{2}-1)^{k-1}}{2\sqrt{2}} \tag{2.9}
\end{aligned}$$

$k = 2m + 1$, $m = 2t$, thus we can substitute $k = 4t + 1$, and using the binomial formula we have

$$\begin{aligned} x + y &= \frac{1}{2\sqrt{2}} \left[\sum_{i=0}^{4t+2} \binom{4t+2}{i} 2^{\frac{i}{2}} - \sum_{i=0}^{4t+2} \binom{4t+2}{i} 2^{\frac{i}{2}} (-1)^i \right] \\ &= \frac{1}{\sqrt{2}} \sum_{i \text{ is odd}} \binom{4t+2}{i} 2^{\frac{i}{2}} = \sum_{i \text{ is odd}} \binom{4t+2}{i} 2^{\frac{i-1}{2}} \end{aligned}$$

therefore,

$$4(x + y) + 1 = 1 + \sum_{i \text{ is odd}} \binom{4t+2}{i} 2^{\frac{i+3}{2}}$$

or denoting $i = 2j + 1$ we can write

$$c^2 = 1 + \sum_{j=0}^{2t} \binom{4t+2}{2j+1} 2^{j+2}$$

We get the same result in the second case, too:

$$\begin{aligned} x - y &= \frac{1}{2\sqrt{2}} \left[\sum_{i=0}^{4t} \binom{4t}{i} 2^{\frac{i}{2}} - \sum_{i=0}^{4t} \binom{4t}{i} 2^{\frac{i}{2}} (-1)^i \right] \\ &= \frac{1}{\sqrt{2}} \sum_{i \text{ is odd}} \binom{4t}{i} 2^{\frac{i}{2}} = \sum_{i \text{ is odd}} \binom{4t}{i} 2^{\frac{i-1}{2}} \end{aligned}$$

therefore,

$$4(x - y) + 1 = 1 + \sum_{i \text{ is odd}} \binom{4t}{i} 2^{\frac{i+3}{2}}$$

or denoting $i = 2j + 1$ we can write:

$$d^2 = 1 + \sum_{j=0}^{2t-1} \binom{4t}{2j+1} 2^{j+2}.$$

We examine a few first values of t :

- for $t = 0$ we have $c^2 = 9, d^2 = 1$;
- for $t = 1$ we have $c^2 = 281, d^2 = 49$, contradiction for c ;
- for $t = 2$ we have $c^2 = 9513, d^2 = 1633$, contradiction.
- for $t = 3$ we have $c^2 = 323129, d^2 = 55441$, contradiction.

Now from (2.8), (2.9) and $k = 4t + 1$ we get

$$\begin{aligned}
c^2 = 1 + 4(x + y) &= 1 + 4\left(\frac{(1 + \sqrt{2})^{4t+2} - (1 - \sqrt{2})^{4t+2}}{2\sqrt{2}}\right) \\
&= 1 + \sqrt{2}[(1 + \sqrt{2})^{2t+1} - (1 - \sqrt{2})^{2t+1}] \\
&= 1 + \sqrt{2}[(3 + 2\sqrt{2})^{2t+1} - (3 - 2\sqrt{2})^{2t+1}]
\end{aligned}$$

$$\begin{aligned}
d^2 = 1 + 4(x - y) &= 1 + 4\left(\frac{(\sqrt{2} + 1)^{4t} - (\sqrt{2} - 1)^{4t}}{2\sqrt{2}}\right) \\
&= 1 + \sqrt{2}[(\sqrt{2} + 1)^{2t} - (\sqrt{2} - 1)^{2t}] \\
&= 1 + \sqrt{2}[(3 + 2\sqrt{2})^{2t} - (3 - 2\sqrt{2})^{2t}] \\
&= 1 + \sqrt{2}[(17 + 12\sqrt{2})^t - (17 - 12\sqrt{2})^t]
\end{aligned}$$

Using these expressions above we build the following Table 2.3 for several t : (recall that $k = 4t + 1$, where k is the exponent in the expression for x and y).

Table 2.3: nonexistence of 2-perfect codes in $J(2w, w)$ for $n < 2.5 * 10^{15}$

t	$1 + 4(x - y)$	$1 + 4(x + y)$	x	$w = \frac{x+3}{2}$
0	1	9	1	2
1	49	281	41	22($\neq 2(12)$)
2	1633	9513	1393	.
3	55441	323129	47321	.
4	1883329	10976841	1607521	.
5	63977713	372889433	.	.
6	2173358881	12667263849	.	.
7	73830224209	430314081401	.	.
8	2508054264193	14618011503753	.	1070379110498
9	85200014758321	496582077046169	.	36361380737782
10	2894292447518689	16869172608065961	.	1235216565974042

Therefore, at least for $n < 2.5 * 10^{15}$, the necessary condition is not satisfied.

Conclusion: from the fact that two roots in the expression for a code strength must be integers and the fact that $x \equiv 1(4)$, we prove that there is no 2-perfect code in $J(n, w)$ where $n = 2w$, for $n < 2.5 * 10^{15}$.

In summary, we proved the following theorem:

Theorem 50. If 2-perfect code C exists in $J(2w, w)$ then

1. $w = \frac{(1+\sqrt{2})^{4t+1} + (1-\sqrt{2})^{4t+1} + 6}{4}$, for some integer t .
2. $\sum_{j \text{ is even}} \binom{4t+1}{2j} - \sum_{j \text{ is odd}} \binom{4t+1}{2j} \equiv 0 \pmod{3}$.
3. $1 + \sum_{j=0}^{2t} \binom{4t+2}{2j+1} 2^{j+2} = 1 + \sqrt{2}[(3 + 2\sqrt{2})^{2t+1} - (3 - 2\sqrt{2})^{2t+1}]$ must be square of integer, if the strength of C is $\sqrt{8w - 11 + 4\sqrt{5 - 6w + 2w^2}}$.
4. $1 + \sum_{j=0}^{2t-1} \binom{4t}{2j+1} 2^{j+2} = 1 + \sqrt{2}[(17 + 12\sqrt{2})^t - (17 - 12\sqrt{2})^t]$ must be square of integer, if the strength of C is $\sqrt{8w - 11 - 4\sqrt{5 - 6w + 2w^2}}$.

□

2.2.4 Binomial moments for e -perfect code in $J(2w, w)$.

In this section we obtain the expression for k -th binomial moments with respect to the difference configuration distributions for $k \geq \varphi + 1$, where φ is the strength of an e -perfect code in $J(2w, w)$.

Let C be an e -perfect code in $J(2w, w)$. Let $\{\alpha_1, \alpha_2\}$ be a partition of N such that $|\alpha_1| = w$, $|\alpha_2| = w$, and a vector of $(w, 0)$ configuration be a codeword. Let A_i be the number of codewords from configuration $(w - i, i)$.

Let B_i be the number of codewords from configuration $(w - i, i)$ in the translate with translate-leader $(w - 1, 1)$.

Let $\{H_1, H_2, H_3, H_4\}$ be a partition of coordinate set N with $|H_1| = |H_4| = w - 1$, $|H_2| = |H_3| = 1$ such that $H_1 \cup H_2 = \alpha_1$ and $H_3 \cup H_4 = \alpha_2$, and let

- $A_{01}^i = \frac{i^2}{w^2} A_i$ the number of codewords from configuration $(w - i, 0, 1, i - 1)$,
- $A_{10}^i = \frac{(w-i)^2}{w^2} A_i$ the number of codewords from configuration $(w - i - 1, 1, 0, i)$,
- $A_{00}^i = \frac{(w-i)i}{w^2} A_i$ the number of codewords from configuration $(w - i, 0, 0, i)$,

- $A_{11}^i = \frac{(w-i)i}{w^2} A_i$ the number of codewords from configuration $(w-i-1, 1, 1, i-1)$.

Note that

$$A_i = A_{01}^i + A_{10}^i + A_{00}^i + A_{11}^i$$

$$B_i = A_{10}^{i-1} + A_{01}^{i+1} + A_{00}^i + A_{11}^i$$

Let φ be the strength of the code. By Lemma 44 for $k \leq \varphi$ we have:

$$\sum_{i=0}^w \binom{i}{k} A_i = \sum_{i=0}^w \binom{i}{k} B_i = \binom{w}{k} \frac{\binom{n-k}{w-k}}{\Phi_e(n, w)} = \frac{|C| \binom{w}{k}}{\binom{n}{k}} \binom{w}{k}.$$

Theorem 51. If C is an e -perfect code in $J(2w, w)$ and φ is its strength, then for each k , $\varphi < k \leq w$, we have

$$w^2 \sum_{i=0}^w \binom{i}{k} \Delta_i = (2wk - k^2 + k) \sum_{i=0}^w \binom{i}{k} A_i - (w - k + 1)^2 \sum_{i=0}^w \binom{i}{k-1} A_i.$$

Proof. For $k \geq \varphi + 1$ we have:

$$\begin{aligned} |C| \binom{w}{k} &= 2 \sum_{i=0}^w \binom{i}{k} A_i + X, \\ |C| \binom{w}{k} &= 2 \sum_{i=0}^w \binom{i}{k} B_i + Y, \end{aligned}$$

where left part of the equations is the number of ways to choose k columns, and the first summand of the right part is the number of ways to choose k columns in only one part of w coordinates, and the second summand of the right part is the number of ways to choose k columns which appear in more than one part of w coordinates.

$$\begin{aligned} X &= \sum_{i=0}^w (X_{01}^i + X_{10}^i + X_{00}^i + X_{11}^i) \\ Y &= \sum_{i=0}^w (Y_{01}^i + Y_{10}^i + Y_{00}^i + Y_{11}^i) \end{aligned}$$

where

$$\begin{aligned}
X_{lj}^i &= \left[\binom{w}{k} - \binom{i}{k} - \binom{w-i}{k} \right] A_{lj}^i \\
Y_{01}^i &= \left[\binom{w}{k} - \binom{i-1}{k} - \binom{w-i+1}{k} \right] A_{01}^i \\
Y_{10}^i &= \left[\binom{w}{k} - \binom{i+1}{k} - \binom{w-i-1}{k} \right] A_{10}^i \\
Y_{00}^i &= X_{00}^i \\
Y_{11}^i &= X_{11}^i
\end{aligned}$$

since $\Delta_i = A_i - B_i$, we get:

$$\begin{aligned}
0 &= 2 \sum_{i=0}^w \binom{i}{k} \Delta_i + \sum_{i=0}^w \left[\binom{w}{k} - \binom{i}{k} - \binom{w-i}{k} \right] A_{01}^i + \sum_{i=0}^w \left[\binom{w}{k} - \binom{i}{k} - \binom{w-i}{k} \right] A_{10}^i \\
&\quad - \sum_{i=0}^w \left[\binom{w}{k} - \binom{i-1}{k} - \binom{w-i+1}{k} \right] A_{01}^i - \sum_{i=0}^w \left[\binom{w}{k} - \binom{i+1}{k} - \binom{w-i-1}{k} \right] A_{10}^i
\end{aligned}$$

We substitute the expressions for A_{01}^i and A_{10}^i :

$$\begin{aligned}
0 &= 2 \sum_{i=0}^w \binom{i}{k} \Delta_i + \sum_{i=0}^w \left[\binom{i-1}{k} + \binom{w-i+1}{k} - \binom{i}{k} - \binom{w-i}{k} \right] \frac{i^2}{w^2} A_i \\
&\quad + \sum_{i=0}^w \left[\binom{i+1}{k} + \binom{w-i-1}{k} - \binom{i}{k} - \binom{w-i}{k} \right] \frac{(w-i)^2}{w^2} A_i,
\end{aligned}$$

or

$$\begin{aligned}
2w^2 \sum_{i=0}^w \binom{i}{k} \Delta_i &= \sum_{i=0}^w \left[\binom{w-i-1}{k-1} (w-i)^2 - \binom{w-i}{k-1} i^2 \right. \\
&\quad \left. + \binom{i-1}{k-1} i^2 - \binom{i}{k-1} (w-i)^2 \right] A_i
\end{aligned}$$

Using the fact that the code is self-complement, we prove the following proposition (see Appendix C).

Proposition 52.

$$\sum_{i=0}^w \left[\binom{w-i-1}{k-1} (w-i)^2 - \binom{w-i}{k-1} i^2 + \binom{i-1}{k-1} i^2 - \binom{i}{k-1} (w-i)^2 \right] A_i =$$

$$= 2(2wk - k^2 + k) \sum_{i=0}^w \binom{i}{k} A_i - 2(w - (k - 1))^2 \sum_{i=0}^w \binom{i}{k-1} A_i.$$

Therefore, we have that

$$w^2 \sum_{i=0}^w \binom{i}{k} \Delta_i = (2wk - k^2 + k) \sum_{i=0}^w \binom{i}{k} A_i - (w - (k - 1))^2 \sum_{i=0}^w \binom{i}{k-1} A_i.$$

□

Chapter 3

Perfect doubly constant weight codes

Constant weight codes are building blocks for general codes in Hamming metric. Similarly, doubly constant weight are building blocks for codes in Johnson metric. Doubly constant weight codes play an important role in obtaining bounds on the sizes of constant weight codes. A natural question is whether there exist perfect doubly constant weight codes.

In this chapter we discuss three types of trivial perfect doubly constant weight codes, show some properties of perfect doubly constant weight codes, construct the family of parameters for codes whose sphere divides the size of whole space (while in Johnson graph we do not know codes with such parameters), and present the necessary condition for existence of an e-perfect code, which is equivalent to Roos' bound in Johnson graph.

3.1 Definitions and properties of perfect doubly constant weight codes

Given five integers, n_1, n_2, w_1, w_2 and d , such that $0 \leq w_1 \leq n_1$ and $0 \leq w_2 \leq n_2$, define *doubly constant weight code* (w_1, n_1, w_2, n_2, d) be a constant weight code of length $n_1 + n_2$ and weight $w_1 + w_2$, with w_1 ones in the first n_1 positions and w_2 ones in the last n_2 positions, and minimum distance d . Note, that because this definition is based on the definition of constant weight codes, the distance d denotes J-distance, as before.

Let $T(w_1, n_1, w_2, n_2, \delta)$ denote the maximum number of codewords in a (w_1, n_1, w_2, n_2, d) code, where $\delta = 2d$ is a H-distance. Upper bounds on $T(w_1, n_1, w_2, n_2, \delta)$ were found and used in [8] to find upper bounds on $A(n, \delta', w)$.

We denote as $V_{w_1, w_2}^{n_1, n_2}$ the space of all binary vectors of length $n_1 + n_2$ and weight

$w_1 + w_2$, with w_1 ones in the first n_1 positions and w_2 ones in the last n_2 positions.

A doubly constant weight code C is called an e -perfect code, if the e -spheres of all the codewords of C form a partition of $V_{w_1, w_2}^{n_1, n_2}$.

The number of codewords of an e -perfect code $C = (w_1, n_1, w_2, n_2, d)$ is

$$|C| = \frac{\binom{n_1}{w_1} \binom{n_2}{w_2}}{\Phi_e(n_1, w_1, n_2, w_2)}$$

where

$$\Phi_e(n_1, w_1, n_2, w_2) = \sum_{i=0}^e \sum_{j=0}^{e-i} \binom{w_1}{i} \binom{n_1 - w_1}{i} \binom{w_2}{j} \binom{n_2 - w_2}{j},$$

and hence we have that

$$\Phi_e(n_1, w_1, n_2, w_2) \mid \binom{n_1}{w_1} \binom{n_2}{w_2}. \quad (3.1)$$

There are some trivial perfect doubly constant weight codes:

1. $V_{w_1, w_2}^{n_1, n_2}$ is 0-perfect.
2. Any $\{v\}$, $v \in V_{w_1, w_2}^{n_1, n_2}$, is $(w_1 + w_2)$ -perfect.
3. If $n_1 = 2w_1$, $n_2 = 2w_2$ and $w_1 + w_2$ is odd, then any pair of vectors with disjoint $w_1 + w_2$ sets of ones (with w_1 ones in the first n_1 positions and w_2 ones in the last n_2 positions) is e -perfect with $e = \frac{w_1 + w_2 - 1}{2}$.

Lemma 53. If C is an e -perfect doubly constant weight code then its minimum J-distance is $2e + 1$.

Proof. Since C is an e -perfect code, it follows that the e -spheres of two codewords with J-distance less than $2e + 1$ have nonempty intersection. Hence, the minimum J-distance of the code is $2e + 1$. □

Lemma 54. If C is an e -perfect doubly constant weight code then $T(w_1, n_1, w_2, n_2, 4e + 2) = |C|$.

Proof. Assume C is an e -perfect doubly constant weight code, then by Lemma 53, it is $(w_1, n_1, w_2, n_2, 2e + 1)$ code and hence the e -spheres around its codewords are disjoint. Since all e -spheres have the same size and they form partition of $V_{w_1, w_2}^{n_1, n_2}$, then $T(w_1, n_1, w_2, n_2, 4e + 2) = |C|$. □

Lemma 55. If $C = (w_1, n_1, w_2, n_2, 2e + 1)$ is an e -perfect doubly constant weight code then the complement of C in the first n_1 positions is an e -perfect code $(n_1 - w_1, n_1, w_2, n_2, 2e + 1)$.

Proof. The Lemma follows from the fact that there exists an isomorphism between the space of all binary vectors of length $n_1 + n_2$ and weight $w_1 + w_2$, with w_1 ones in the first n_1 positions and w_2 ones in the last n_2 positions and its complement in the first n_1 positions. □

Corollary 56. If $C = (w_1, n_1, w_2, n_2, 2e + 1)$ is an e -perfect doubly constant weight code then the complement of C in the last n_2 positions is an e -perfect code $(w_1, n_1, n_2 - w_2, n_2, 2e + 1)$.

Corollary 57. If $C = (w_1, n_1, w_2, n_2, 2e + 1)$ is an e -perfect doubly constant weight code then the complement of C is an e -perfect code $(n_1 - w_1, n_1, n_2 - w_2, n_2, 2e + 1)$.

From Lemma 53, Lemma 55, Corollary 56 and Corollary 57 follows:

Corollary 58. If $C = (w_1, n_1, w_2, n_2, 2e + 1)$ is a non trivial e -perfect doubly constant weight code then $w_1 + w_2 \geq 2e + 1$, $n_1 + n_2 - w_1 - w_2 \geq 2e + 1$, $n_1 - w_1 + w_2 \geq 2e + 1$ and $w_1 + n_2 - w_2 \geq 2e + 1$.

3.2 Family of parameters for codes whose size of sphere, $\Phi_1(n_1, w_1, n_2, w_2)$, divides the size of whole space

In this section we show the family of parameters for codes that satisfy the necessary condition (3.1) for existence a 1-perfect doubly constant weight code.

Proposition 59. Let k be a natural number and C be a doubly constant weight code $(w_1, n_1, w_2, n_2, 3)$, when $w_1 = w_2 = 2k$, $n_1 = 4k + 1$, and $n_2 = 4k + 2$. Then $\Phi_1(n_1, w_1, n_2, w_2) \mid \binom{n_1}{w_1} \binom{n_2}{w_2}$.

Proof.

$$\begin{aligned} \Phi_1(n_1, w_1, n_2, w_2) &= 1 + w_1(n_1 - w_1) + w_2(n_2 - w_2) \\ &= 1 + 2k(2k + 1) + 2k(2k + 2) = (2k + 1)(4k + 1), \end{aligned}$$

therefore we have to prove that

$$\frac{\binom{4k+1}{2k} \binom{4k+2}{2k}}{(2k+1)(4k+1)} \in \mathbb{Z}.$$

But

$$\frac{\binom{4k+1}{2k}}{4k+1} = \binom{4k}{2k} \frac{1}{2k+1} \in \mathbb{Z}$$

is a Catalan number [24] and

$$\frac{\binom{4k+2}{2k}}{2k+1} = \binom{4k+2}{2k+1} \frac{1}{2k+2} \in \mathbb{Z}$$

is also Catalan number, hence

$$\Phi_1(4k+1, 2k, 4k+2, 2k) \mid \binom{4k+1}{2k} \binom{4k+2}{2k}.$$

□

Codes with parameters as above are candidates for being perfect codes. But from [25] we can see that for small k ($k = 1, 2, \text{or } 3$) there are no 1-perfect doubly constant weight codes with such parameters. Still, we can not say anything about the codes with higher values of k .

3.3 Necessary condition for existence of an e -perfect doubly constant weight code

In this section we prove the theorem that gives the bound for parameters of e -perfect code. This bound is similar to the Roos's bound in Johnson graph. Hence, the techniques that we use here are a generalization of the ideas of the proof of Roos' bound by Etzion [13].

We recall a few definitions which we will use in the proof of the existence theorem.

For a given partition of set of all $n_1 + n_2$ coordinates into four subsets α, β, γ and δ , let *configuration* (a, b, c, d) be a set of all vectors with weight a in the positions of α , weight b in the positions of β , weight c in the positions of γ and weight d in the positions of δ .

For an e -perfect doubly constant weight code C we say that $w \in C$ *J-cover* $v \in V_{w_1, w_2}^{n_1, n_2}$ if the J-distance between u and v less or equal to e .

Theorem 60 . If an e -perfect doubly constant weight code $(w_1, n_1, w_2, n_2, 2e+1)$ exists then

$$n_1 \leq \frac{(2e+1)(w_1-1) + w_2}{e}$$

and

$$n_2 \leq \frac{(2e+1)(w_2-1) + w_1}{e}.$$

Proof. Assume C is an e -perfect code $(w_1, n_1, w_2, n_2, 2e+1)$.

Case 1: $w_1 > e$

We partition the set of coordinates into four subsets α, β, γ and δ such that $|\alpha| = w_1 - 1$, $|\beta| = w_2$, $|\gamma| = n_1 - w_1 + 1$, $|\delta| = n_2 - w_2$, and there is a codeword of configuration $(w_1 - (e+1), w_2, e+1, 0)$. The J-distance between a vector from configuration $(w_1 - (e+1), w_2, e+1, 0)$ and a vector from configuration $(w_1 - a, w_2 - b, a, b)$, $0 < a + b \leq e$, is strictly less than $2e+1$, so C does not have any codeword from configuration $(w_1 - a, w_2 - b, a, b)$, $0 < a + b \leq e$. Therefore, all the vectors from configuration $(w_1 - 1, w_2, 1, 0)$ are J-covered by codewords from configuration $(w_1 - (e+1), w_2, e+1, 0)$, or $(w_1 - e, w_2 - 1, e, 1)$, or $(w_1 - (e-1), w_2 - 2, e-1, 2), \dots$, or $(w_1 - 1, w_2 - e, 1, e)$.

Let X_i , $0 \leq i \leq e$, be a collection of codewords from configuration $(w_1 - (e+1-i), w_2 - i, e+1-i, i)$, such that $\bigcup_{i=0}^e X_i$ J-cover all the vectors from configuration $(w_1 - 1, w_2, 1, 0)$. There are $n_1 - w_1 + 1$ vectors from configuration $(w_1 - 1, w_2, 1, 0)$ and each codeword in X_i J-covers $e+1-i$ such vectors. Therefore,

$$\sum_{i=0}^e (e+1-i)|X_i| = n_1 - w_1 + 1. \quad (3.2)$$

Since the minimum J-distance is $2e+1$, two codewords in $\bigcup_{i=0}^{e-1} X_i$ cannot intersect in the zeroes of part α , and two codewords in $\bigcup_{i=1}^e X_i$ cannot intersect in the zeroes of part β . Hence,

$$\sum_{i=0}^{e-1} (e-i)|X_i| \leq w_1 - 1 \quad (3.3)$$

$$\sum_{i=1}^e i|X_i| \leq w_2. \quad (3.4)$$

Since

$$\frac{e+1}{e} \sum_{i=0}^{e-1} (e-i)|X_i| + \frac{1}{e} \sum_{i=1}^e i|X_i| = \sum_{i=0}^e (e+1-i)|X_i|,$$

from (3.2), (3.3) and (3.4) above follows:

$$n_1 - w_1 + 1 = \sum_{i=0}^e (e+1-i)|X_i| = \frac{e+1}{e} \sum_{i=0}^{e-1} (e-i)|X_i| + \frac{1}{e} \sum_{i=1}^e i|X_i| \leq \frac{e+1}{e} (w_1 - 1) + \frac{1}{e} w_2$$

Therefore,

$$n_1 \leq (w_1 - 1)\left(\frac{e+1}{e} + 1\right) + \frac{w_2}{e} = \frac{(2e+1)(w_1-1) + w_2}{e}.$$

Case 2: $1 < w_1 \leq e$.

Let $w_1 = e - k$ for some k , $0 \leq k < e - 1$.

We use the same partition as in the Case 1: we partition the set of coordinates into four subsets α, β, γ and δ such that $|\alpha| = w_1 - 1 = e - k - 1$, $|\beta| = w_2$, $|\gamma| = n_1 - w_1 + 1 = n_1 - e + k + 1$, $|\delta| = n_2 - w_2$, and there is a codeword of configuration $(0, w_2 - k - 1, e - k, k + 1)$. All the vectors from configuration $(e - k - 1, w_2, 1, 0)$ are J-covered only by codewords from configuration $(0, w_2 - k - 1, e - k, k + 1)$ or from configuration $(e - k - 1, w_2 - e, 1, e)$, because of the restriction on minimal distance $2e + 1$.

Let X be a set of codewords from configuration $(0, w_2 - k - 1, e - k, k + 1)$ and Y a set of codewords from configuration $(e - k - 1, w_2 - e, 1, e)$, such that codewords in $X \cup Y$ cover all the vectors from configuration $(e - k - 1, w_2, 1, 0)$. Therefore,

$$(e - k)|X| + |Y| = n_1 - e + k + 1. \quad (3.5)$$

Note, that the J-distance between two codewords in X less or equal then $e + k + 2$. As $e - 1 > k$, it follows that

$$|X| \leq 1, \quad (3.6)$$

Since the minimum J-distance is $2e + 1$, two codewords in $X \cup Y$ cannot intersect in the zeroes of part β . Hence,

$$|Y|e \leq w_2 - k - 1, \quad (3.7)$$

From (3.5), (3.6) and (3.7) follows

$$n_1 - e + k + 1 \leq e - k + \frac{w_2 - k - 1}{e}.$$

Thus,

$$\begin{aligned} n_1 &\leq e - k - 1 + e - k + \frac{w_2 - k - 1}{e} = \frac{(2e - 2k - 1)e + w_2 - k - 1}{e} \\ &= \frac{2e^2 - 2ke - e + w_2 - k - 1}{e} = \frac{(2e + 1)(e - k - 1) + w_2}{e}. \end{aligned}$$

Case 3: $w_1 = 1$.

Now our partition is as follows: $|\alpha| = 0$, $|\beta| = w_2$, $|\gamma| = n_1$, $|\delta| = n_2 - w_2$ and there is a codeword of configuration $(0, w_2 - e, 1, e)$. Let X be a set of codewords from

configuration $(0, w_2 - e, 1, e)$. Hence, all the vectors from configuration $(0, w_2, 1, 0)$ are J -covered by the codewords from X . In addition, two codewords in X cannot intersect in the zeroes part β . Therefore,

$$n_1 = |X| \leq \frac{w_2}{e} = \frac{(2e+1)(w_1-1) + w_2}{e}.$$

As we can swap the roles of n_1 and n_2 , and w_1 and w_2 we obtain the bound on n_2 :

$$n_2 \leq \frac{(2e+1)(w_2-1) + w_1}{e}.$$

□

Chapter 4

Steiner Systems and doubly Steiner Systems

There is tight connection between constant weight codes and Steiner systems, and doubly constant weight codes and doubly Steiner systems. As an example of such connections, observe Steiner systems which are optimal constant weight codes and doubly Steiner Systems which are optimal doubly weight codes [26].

This chapter is organized as follows. In Section 4.1 we give definitions and theorems that will be used in the following sections. In Section 4.2 we prove the bound on the length of Steiner system using anticodes. In Section 4.3 we consider the doubly Steiner system and get analogous results in this structure.

4.1 Definitions and known results

Let us recall the definition of Steiner systems.

A Steiner System $S(t, w, n)$ is a collection of w -subsets (called blocks) taken from an n -set such that each t -subset of the n -set is contained in exactly one block.

If we represent blocks as 0-1 -vectors we observe that a Steiner system $S(t, w, n)$ is equivalent to a constant weight code with parameters $(n, 2(w - t + 1), w)$, since any two vectors have at most $t - 1$ ones in common.

Steiner systems play an important role in ruling out the existence of e -perfect codes in $J(n, w)$. Moreover, the Steiner systems $S(1, w, 2w)$, where w is odd, and $S(w, w, n)$, are among the trivial perfect codes in the Johnson graph. Etzion proved that there are no more Steiner systems which are also perfect codes in the Johnson graph [15].

We remind a few definitions which we will use in the following.

A connected graph Γ with diameter d is called *distance-regular* if for any vertices x and y of Γ and any integers $0 \leq i, j \leq d$, the number of vertices z at distance i from x and at distance j from y depends only on i, j and $k := \text{dist}(x, y)$ and not on the choice of x and y themselves.

The following theorem is due to Delsarte[6]:

Theorem 61 : Let X and Y be subsets of the vertex set V of a distance regular graph Γ , such that nonzero distances occurring between vertex in X do not occur between vertices of Y . Then $|X| \cdot |Y| \leq |V|$.

A subset X of V is called an *anticode* with diameter D , if D is the maximum distance occurring between vertices of X .

Anticodes with diameter D having maximal size are called *optimal anticodes*.

Ahlsweede, Aydinian and Khachatrian [19] gave a new definition of diameter-perfect codes (D -perfect codes). They examined a variant of Theorem 61.

Let Γ be a distance-regular graph with a vertex set V . If A is an anticode in Γ , denote by $D(A)$ the diameter of A . Let $A^*(D) = \max \{|A| : D(A) \leq D\}$.

Theorem 62 [19] . If C is a code in Γ with minimum distance $D + 1$, then $|C| \leq |V| \cdot |A^*(D^{-1})|$.

A code C with minimum distance $D + 1$ is called D -perfect if Theorem 62 holds with equality. This is a generalization of the usual definition of e -perfect codes as e -spheres are anticodes with diameter $2e$.

Lemma 63 [19]. Any Steiner system $S(t, w, n)$ forms a diameter perfect code.

We show the proof from [19] for completeness, since we use it in the next section.

Proof. Let C be an $(n, 2(w - t + 1), w)$ -code corresponding to a $S(t, w, n)$. Then

$$|C| = \frac{\binom{n}{t}}{\binom{w}{t}} = \frac{\binom{n}{w}}{\binom{n-t}{w-t}}.$$

On the other hand $|C| \leq \frac{\binom{n}{w}}{|A^*(n, 2(w-t), w)|}$, where $A^*(n, 2(w-t), w)$ is an optimal anticode in $J(n, w)$ of diameter $2(w-t)$ (H-distance). Therefore $|A^*(n, 2(w-t), w)| \leq \binom{n-t}{w-t}$. Since there exists an anticode of size $\binom{n-t}{w-t}$ the statement follows. \square

4.2 Necessary condition for existence of Steiner system

In this section we provide an anticode-based proof of the bound on Steiner system, which is different from the existing proof of Tits [27]. We note that similar two techniques were

used to prove Roos's bound, one by Roos [12] based on anticode and the Theorem 61 of Delsarte, and another one by Etzion [13] based on specific partition of set of coordinates and J-covering some vectors by codeword of specific configuration.

We first mention the proof by Tits for completeness.

Theorem 64. If Steiner System $S(t, w, n)$ exists with $w < n$ then

$$n \geq (t+1)(w-t+1).$$

Proof 1 (Tits 1964 [27]):

Let T be a $t+1$ -subset of the n -set, such that $T \not\subseteq B$, for all blocks B . Such a $t+1$ -set T exists. There exactly $t+1$ blocks B_0, \dots, B_t with $|B_i \cap T| = t$ ($i = 0, \dots, t$). The point sets $B_i \setminus T$ are mutually disjoint. Hence

$$n \geq |T| + \sum_{i=0}^t |B_i \setminus T| = (w-t+1)(t+1)$$

□

Proof 2 (based on anticode):

Assume $S(t, w, n)$ exists. Then by Lemma 63 for any anticode $A(n, w-t, w)$ in $J(n, w)$ with diameter $w-t$ (J-distance) we have

$$A(n, w-t, w) \leq \binom{n-t}{w-t}, \quad (4.1)$$

since we know that there is an optimal anticode with diameter $w-t$ and size $\binom{n-t}{w-t}$.

We will construct an anticode with diameter $w-t$ for Steiner system $S(t, w, n)$.

Let S be a set of coordinates of size $t+2$. Denote A_t to be a collection of sets of coordinates of size w which intersects the given set S in at least $t+1$ coordinates. We get the anticode with diameter $w-t$ and size $\binom{n-t-2}{w-t-2} + (t+2)\binom{n-t-2}{w-t-1}$. From (4.1) we have

$$\binom{n-t-2}{w-t-2} + (t+2)\binom{n-t-2}{w-t-1} \leq \binom{n-t}{w-t},$$

or

$$n \geq (t+1)(w-t+1).$$

□

4.3 Doubly Steiner system

We start this section with new definitions :

A $(w_1, n_1, w_2, n_2, d = w_1 + w_2 - t_1 - t_2 + 1)$ code is *perfect (t_1, t_2) cover* if every word from configuration (t_1, t_2) is contained in exactly one codeword. Note, that all the codewords are from configuration (w_1, w_2) . The definition of doubly constant weight code which is a perfect cover is akin to a constant weight code which is a Steiner system. Hence, one can call such a code *doubly Steiner system* $S(t_1, t_2, w_1, w_2, n_1, n_2)$.

In [26] Etzion show that a doubly Steiner system $S(t_1, t_2, w_1, w_2, n_1, n_2)$ is an optimal $(w_1, n_1, w_2, n_2, 1 + w_2 - t_1 - t_2 + 1)$ code, and present the bounds on the length of such code.

In the follows we prove that the doubly Steiner system is a diameter perfect code and present the new bound on its length, equivalent to bound of Tits for Steiner system.

Lemma 65. Any doubly Steiner system $S(t_1, t_2, w_1, w_2, n_1, n_2)$ forms a diameter perfect code.

Proof. Let C be a $(w_1, n_1, w_2, n_2, (w_1 + w_2 - t_1 - t_2 + 1))$ code which is a perfect (t_1, t_2) -cover corresponding to a $S(t_1, t_2, w_1, w_2, n_1, n_2)$. Then

$$|C| = \frac{\binom{n_1}{t_1} \binom{n_2}{t_2}}{\binom{w_1}{t_1} \binom{w_2}{t_2}} = \frac{\binom{n_1}{w_1} \binom{n_2}{w_2}}{\binom{n_1 - t_1}{w_1 - t_1} \binom{n_2 - t_2}{w_2 - t_2}}.$$

On the other hand by Theorem 62, $|C| \leq \frac{\binom{n_1}{w_1} \binom{n_2}{w_2}}{A^*(w_1, n_1, w_2, n_2, (w_1 + w_2 - t_1 - t_2))}$, where $A^*(w_1, n_1, w_2, n_2, (w_1 + w_2 - t_1 - t_2))$ is an optimal anticode with diameter $(w_1 + w_2 - t_1 - t_2)$. Therefore

$$A^*(w_1, n_1, w_2, n_2, (w_1 + w_2 - t_1 - t_2)) \leq \binom{n_1 - t_1}{w_1 - t_1} \binom{n_2 - t_2}{w_2 - t_2}.$$

We construct an anticode of size $\binom{n_1 - t_1}{w_1 - t_1} \binom{n_2 - t_2}{w_2 - t_2}$ as follows. We take a constant set of coordinates of size t_1 in the first n_1 coordinates and t_2 in the last n_2 coordinates and complete it by all vectors of size $w_1 - t_1$ in the first part and $w_2 - t_2$ in the last part .

Since there exists an anticode of size $\binom{n_1 - t_1}{w_1 - t_1} \binom{n_2 - t_2}{w_2 - t_2}$, the statement follows. □

Corollary 66.

For any anticode $A(w_1, n_1, w_2, n_2, (w_1 + w_2 - t_1 - t_2))$ with diameter $w_1 + w_2 - t_1 - t_2$ (Johnson distance) we have

$$A(w_1, n_1, w_2, n_2, (w_1 + w_2 - t_1 - t_2)) \leq \binom{n_1 - t_1}{w_1 - t_1} \binom{n_2 - t_2}{w_2 - t_2}.$$

Theorem 67. If a doubly Steiner system $S(t_1, t_2, w_1, w_2, n_1, n_2)$ exists and $t_2 > t_1, t_1 < w_1$ then

$$n_1 \geq (t_1 + 1)w_1 - t_1 t_2$$

$$n_2 \geq (t_2 + 1)(w_2 - t_2 + 1).$$

Proof. Let C be a $(w_1, n_1, w_2, n_2, (w_1 + w_2 - t_1 - t_2 + 1))$ code which is a perfect (t_1, t_2) -cover, corresponding to a $S(t_1, t_2, w_1, w_2, n_1, n_2)$.

Let S be a vector from configuration $(t_1 + 1, t_2)$, which is not contained in any codeword. Consider $t_1 + 1$ subvectors of S from configuration (t_1, t_2) . Each of them is contained in exactly one codeword. Since the minimal distance of the code is $w_1 + w_2 - t_1 - t_2 + 1$, there are precisely $t_1 + 1$ codewords which contain those vectors, and these $t_1 + 1$ codewords are disjoint outside of S . Therefore in the first n_1 coordinates we have:

$$n_1 - (t_1 + 1) \geq (w_1 - t_1)(t_1 + 1)$$

or

$$n_1 \geq (w_1 - t_1 + 1)(t_1 + 1)$$

and in the last n_2 coordinates we have:

$$n_2 - t_2 \geq (w_2 - t_2)(t_1 + 1)$$

or

$$n_2 \geq w_2(t_1 + 1) - t_1 t_2.$$

By swapping the roles of n_1 and n_2 , and w_1 and w_2 , we get that

$$n_1 \geq w_1(t_2 + 1) - t_1 t_2.$$

Therefore,

$$n_1 \geq \max\{w_1(t_2 + 1) - t_1 t_2, (w_1 - t_1 + 1)(t_1 + 1)\}$$

$$n_2 \geq \max\{w_2(t_1 + 1) - t_1 t_2, (w_2 - t_2 + 1)(t_2 + 1)\}$$

If we write $t_2 = t_1 + a$, where $a > 0$ is an integer, we can rewrite the last expression as follows:

$$\begin{aligned} & \max\{w_1(t_1 + a + 1) - t_1(t_1 + a), (w_1 - t_1 + 1)(t_1 + 1)\} \\ &= \max\{w_1 t_1 + w_1 - t_1^2 + (w_1 - t_1)a, w_1 t_1 + w_1 - t_1^2 + 1\} \end{aligned}$$

$$= w_1 t_1 + w_1 - t_1^2 + (w_1 - t_1)a$$

Therefore,

$$n_1 \geq w_1(t_2 + 1) - t_1 t_2.$$

Similarly we obtain

$$\max\{w_2(t_1 + 1) - t_1 t_2, (w_2 - t_2 + 1)(t_2 + 1)\}$$

$$= (w_2 - t_2 + 1)(t_2 + 1).$$

Therefore,

$$n_2 \geq (t_2 + 1)(w_2 - t_2 + 1).$$

□

Appendix A

Proof of Proposition 48.

Proposition 48. For all $k, \varphi < k \leq w$, we have

$$0 = [1 + k^2 - k(1 + n) + nw - w^2] \sum_{i=0}^w \binom{i}{k} \Delta_i + (1 - k + w)^2 \sum_{i=0}^w \binom{i}{k-1} \Delta_i$$

Proof. In the Section 2 we saw that for all $k, \varphi < k \leq w$,

$$\begin{aligned} 0 &= \sum_{i=0}^w \binom{i}{k} (i+1)^2 \Delta_{i+1} + (n-2w) \sum_{i=0}^w \binom{i}{k} (i+1) \Delta_{i+1} + \sum_{i=0}^w \binom{i}{k} \Delta_i \\ &+ w(n-2w) \sum_{i=0}^w \binom{i}{k} \Delta_i + (4w-n) \sum_{i=0}^w \binom{i}{k} i \Delta_i - 2 \sum_{i=0}^w \binom{i}{k} i^2 \Delta_i \\ &+ w^2 \sum_{i=0}^w \binom{i}{k} \Delta_{i-1} - 2w \sum_{i=0}^w \binom{i}{k} (i-1) \Delta_{i-1} + \sum_{i=0}^w \binom{i}{k} (i-1)^2 \Delta_{i-1} \end{aligned}$$

Now we simplify it, using follows equations:

$$\begin{aligned} \binom{i}{k} (i+1) &= \binom{i+1}{k+1} (k+1) \\ \binom{i}{k} &= \binom{i-1}{k-1} + \binom{i-1}{k} \\ \binom{i}{k-1} i^2 + (k+1) \binom{i}{k+1} i - \binom{i}{k} i^2 &= i(k-1) \binom{i}{k-1} \end{aligned}$$

$$\binom{i}{k-1}i = k\binom{i}{k} + (k-1)\binom{i}{k-1}$$

$$\begin{aligned} 0 &= (k+1) \sum_{i=0}^w \binom{i+1}{k+1} (i+1)\Delta_{i+1} + (n-2w)(k+1) \sum_{i=0}^w \binom{i+1}{k+1} \Delta_{i+1} \\ &+ \sum_{i=0}^w \binom{i}{k} \Delta_i + w(n-2w) \sum_{i=0}^w \binom{i}{k} \Delta_i + (4w-n) \sum_{i=0}^w \binom{i}{k} i\Delta_i - 2 \sum_{i=0}^w \binom{i}{k} i^2 \Delta_i \\ &+ w^2 \sum_{i=0}^w \binom{i-1}{k-1} \Delta_{i-1} + w^2 \sum_{i=0}^w \binom{i-1}{k} \Delta_{i-1} - 2w \sum_{i=0}^w \binom{i-1}{k-1} (i-1)\Delta_{i-1} \\ &- 2w \sum_{i=0}^w \binom{i-1}{k} (i-1)\Delta_{i-1} + \sum_{i=0}^w \binom{i-1}{k-1} (i-1)^2 \Delta_{i-1} + \sum_{i=0}^w \binom{i-1}{k} (i-1)^2 \Delta_{i-1} \end{aligned}$$

$$\begin{aligned} 0 &= (k+1) \sum_{i=0}^w \binom{i}{k+1} i\Delta_i + (n-2w)(k+1) \sum_{i=0}^w \binom{i}{k+1} \Delta_i + \sum_{i=0}^w \binom{i}{k} \Delta_i \\ &+ w(n-2w) \sum_{i=0}^w \binom{i}{k} \Delta_i + (4w-n) \sum_{i=0}^w \binom{i}{k} i\Delta_i - 2 \sum_{i=0}^w \binom{i}{k} i^2 \Delta_i \\ &+ w^2 \sum_{i=0}^w \binom{i}{k-1} \Delta_i + w^2 \sum_{i=0}^w \binom{i}{k} \Delta_i - 2w \sum_{i=0}^w \binom{i}{k-1} i\Delta_i \\ &- 2w \sum_{i=0}^w \binom{i}{k} i\Delta_i + \sum_{i=0}^w \binom{i}{k-1} i^2 \Delta_i + \sum_{i=0}^w \binom{i}{k} i^2 \Delta_i \end{aligned}$$

$$\begin{aligned} 0 &= \underbrace{(k+1) \sum_{i=0}^w \binom{i}{k+1} i\Delta_i}_{\text{}} + \underbrace{(n-2w)(k+1) \sum_{i=0}^w \binom{i}{k+1} \Delta_i}_{\text{}} \\ &+ \underbrace{(wn-w^2+1) \sum_{i=0}^w \binom{i}{k} \Delta_i + (2w-n) \sum_{i=0}^w \binom{i}{k} i\Delta_i - \sum_{i=0}^w \binom{i}{k} i^2 \Delta_i}_{\text{}} \\ &+ \underbrace{w^2 \sum_{i=0}^w \binom{i}{k-1} \Delta_i - 2w \sum_{i=0}^w \binom{i}{k-1} i\Delta_i + \sum_{i=0}^w \binom{i}{k-1} i^2 \Delta_i}_{\text{}} \end{aligned}$$

$$\begin{aligned}
0 &= \frac{(k-1) \sum_{i=0}^w \binom{i}{k-1} i \Delta_i + (2w-n)k \sum_{i=0}^w \binom{i}{k} \Delta_i + (wn-w^2+1) \sum_{i=0}^w \binom{i}{k} \Delta_i}{\quad \quad \quad} \\
&+ w^2 \sum_{i=0}^w \binom{i}{k-1} \Delta_i - 2w \sum_{i=0}^w \binom{i}{k-1} i \Delta_i \\
&= (k-1-2w) \sum_{i=0}^w \binom{i}{k-1} i \Delta_i + w^2 \sum_{i=0}^w \binom{i}{k-1} \Delta_i \\
&+ [(2w-n)k + wn - w^2 + 1] \sum_{i=0}^w \binom{i}{k} \Delta_i \\
&= [(2w-n)k + wn - w^2 + 1 + k(k-1-2w)] \sum_{i=0}^w \binom{i}{k} \Delta_i \\
&+ [w^2 + (k-1)(k-1-2w)] \sum_{i=0}^w \binom{i}{k-1} \Delta_i
\end{aligned}$$

Finally, we get:

$$0 = [1 + k^2 - k(1+n) + nw - w^2] \sum_{i=0}^w \binom{i}{k} \Delta_i + (1-k+w)^2 \sum_{i=0}^w \binom{i}{k-1} \Delta_i$$

□

Appendix B

Proof of Proposition 49

Proposition 49. For each k , $\varphi < k \leq w$, we have

$$\begin{aligned}
0 &= \frac{1}{4}[4 + k^4 + 5w^2 - 2w^3 + w^4 - 2k^3(1 + 2w) + k^2(7 + 2w + 6w^2) \\
&\quad - 2k(3 + 5w - w^2 + 2w^3)] \sum_{i=0}^w \binom{i}{k} \Delta_i \\
&\quad + \frac{1}{2}(1 - k + w)^2(4 + k^2 + w^2 - 2k(1 + w)) \sum_{i=0}^w \binom{i}{k-1} \Delta_i \\
&\quad + \frac{1}{4}(1 - k + w)^2(2 - k + w)^2 \sum_{i=0}^w \binom{i}{k-2} \Delta_i
\end{aligned}$$

Proof. We use the following identities:

$$\begin{aligned}
\binom{i}{k}(i+1) &= (k+1) \binom{i+1}{k+1} \\
\binom{i}{k} \binom{i+2}{2} &= \frac{(k+1)(k+2)}{2} \binom{i+2}{k+2} \\
\binom{i}{k} &= \binom{i-1}{k-1} + \binom{i-1}{k} = \binom{i-2}{k-2} + 2\binom{i-2}{k-1} + \binom{i-2}{k}
\end{aligned}$$

for the calculations below.

$$\begin{aligned}
0 &= \frac{(k+1)(k+2)}{2} \sum_{i=0}^w \binom{i+2}{k+2} \binom{i+2}{2} \Delta_{i+2} \\
&+ \sum_{i=0}^w \left[\binom{i-2}{k-2} + 2 \binom{i-2}{k-1} + \binom{i-2}{k} \right] \binom{w-(i-2)}{2}^2 \Delta_{i-2} \\
&+ (k+1) \sum_{i=0}^w \binom{i+1}{k+1} \left[(i+1) + 2(w-(i+1)) \binom{i+1}{2} \right] \Delta_{i+1} \\
&+ \sum_{i=0}^w \left[\binom{i-1}{k-1} + \binom{i-1}{k} \right] \\
&* \left[(w-(i-1))^2 + 2(i-1)(w-(i-1)) \binom{w-(i-1)}{2} \right] \Delta_{i-1} \\
&+ \sum_{i=0}^w \binom{i}{k} \left[1 + 2i(w-i) + 2 \binom{i}{2} \binom{w-i}{2} + i^2(w-i)^2 \right] \Delta_i
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{(k+1)(k+2)}{2} \sum_{i=0}^w \binom{i}{k+2} \binom{i}{2} \Delta_i \\
&+ \sum_{i=0}^w \left[\binom{i}{k-2} + 2 \binom{i}{k-1} + \binom{i}{k} \right] \binom{w-i}{2}^2 \Delta_i \\
&+ (k+1) \sum_{i=0}^w \binom{i}{k+1} \left[i + 2(w-i) \binom{i}{2} \right] \Delta_i \\
&+ \sum_{i=0}^w \left[\binom{i}{k-1} + \binom{i}{k} \right] \left[(w-i)^2 + 2i(w-i) \binom{w-i}{2} \right] \Delta_i \\
&+ \sum_{i=0}^w \binom{i}{k} \left[1 + 2i(w-i) + 2 \binom{i}{2} \binom{w-i}{2} + i^2(w-i)^2 \right] \Delta_i
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{(k+1)(k+2)}{4} \sum_{i=0}^w \binom{i}{k+2} (i^2 - i) \Delta_i \\
&+ \frac{1}{4} \sum_{i=0}^w \binom{i}{k-2} (w-i)^2 (w-i-1)^2 \Delta_i \\
&+ \sum_{i=0}^w \binom{i}{k-1} \left[\frac{(w-i)^2 (w-i-1)^2}{2} + [(w-i)^2 + i(w-i)^2 (w-i-1)] \right] \Delta_i \\
&+ (k+1) \sum_{i=0}^w \binom{i}{k+1} [i + (w-i)(i-1)] \Delta_i \\
&+ \sum_{i=0}^w \binom{i}{k} \left[\frac{(w-i)^2 (w-i-1)^2}{4} + (w-i)^2 + i(w-i)^2 (w-i-1) + 1 \right. \\
&\quad \left. + 2i(w-i) + \frac{i(i-1)(w-i)(w-i-1)}{2} + i^2 (w-i)^2 \right] \Delta_i
\end{aligned}$$

Finally we obtain:

$$\begin{aligned}
0 &= \frac{1}{4} (k+1)(k+2) \sum_{i=0}^w \binom{i}{k+2} [i^2 - i] \Delta_i \\
&+ (k+1) \sum_{i=0}^w \binom{i}{k+1} [(1-w)i + (1+w)i^2 - i^3] \Delta_i \\
&+ \sum_{i=0}^w \binom{i}{k} \left[\left(1 + \frac{5w^2}{4} - \frac{w^3}{2} + \frac{w^4}{4}\right) + \left(-\frac{5}{4} + w\right)i^2 + \left(-\frac{1}{2} - w\right)i^3 + \frac{3}{4}i^4 \right] \Delta_i \\
&+ \sum_{i=0}^w \binom{i}{k-1} \left[\left(\frac{3}{2}w^2 - w^3 + \frac{w^4}{2}\right) + (-3w + 2w^2 - w^3)i \right. \\
&\quad \left. + \left(\frac{3}{2} - w\right)i^2 + wi^3 - \frac{1}{2}i^4 \right] \Delta_i \\
&+ \frac{1}{4} \sum_{i=0}^w \binom{i}{k-2} [(w^2 - 2w^3 + w^4) + (-2w + 6w^2 - 4w^3)i \\
&\quad + (1 - 6w + 6w^2)i^2 + (2 - 4w)i^3 + i^4] \Delta_i
\end{aligned}$$

Now we use following identities for computation of coefficients of $\sum_{i=0}^w \binom{i}{j}$ for $j = k-2, \dots, k+4$:

$$\begin{aligned}
\binom{i}{k} i &= (k+1) \binom{i}{k+1} + k \binom{i}{k} \\
\binom{i}{k} i^2 &= (k+1)(k+2) \binom{i}{k+2} + (k+1)(2k+1) \binom{i}{k+1} + k^2 \binom{i}{k} \\
\binom{i}{k} i^3 &= (k+1)(k+2)(k+3) \binom{i}{k+3} + 3(k+1)^2(k+2) \binom{i}{k+2} \\
&\quad + (k+1)(3k^2+3k+1) \binom{i}{k+1} + k^3 \binom{i}{k} \\
\binom{i}{k} i^4 &= (k+1)(k+2)(k+3)(k+4) \binom{i}{k+4} \\
&\quad + (4k+6)(k+1)(k+2)(k+3) \binom{i}{k+3} \\
&\quad + (k+1)(k+2)(6k^2+12k+7) \binom{i}{k+2} \\
&\quad + (k+1)(4k^3+6k^2+4k+1) \binom{i}{k+1} + k^4 \binom{i}{k}
\end{aligned}$$

The coefficient of $\sum_{i=0}^w \binom{i}{k+4} \Delta_i$ follows from $\binom{i}{k} i^4$, $\binom{i}{k+1} i^3$, $\binom{i}{k+2} i^2$, therefore it equals to:

$$\begin{aligned}
&\frac{1}{4}(k+1)(k+2)(k+3)(k+4) + (k+1)(-(k+2)(k+3)(k+4)) \\
&+ \frac{3}{4}(k+1)(k+2)(k+3)(k+4) \\
&= (k+1)(k+2)(k+3)(k+4) \left[\frac{1}{4} - 1 + \frac{3}{4} \right] = 0
\end{aligned}$$

The coefficient of $\sum_{i=0}^w \binom{i}{k+3} \Delta_i$ follows from $\binom{i}{k+2} i^2$, $\binom{i}{k+2} i$, $\binom{i}{k+1} i^2$, $\binom{i}{k+1} i^3$, $\binom{i}{k} i^3$, $\binom{i}{k} i^4$, $\binom{i}{k-1} i^4$, therefore it equals to:

$$\begin{aligned}
& \frac{(k+1)(k+2)}{4}(k+3)(2k+5) - \frac{(k+1)(k+2)}{4}(k+3) \\
+ & (k+1)(1+w)(k+2)(k+3) \\
- & (k+1)3(k+2)^2(k+3) + \left(-\frac{1}{2} - w\right)(k+1)(k+2)(k+3) \\
+ & \frac{3}{4}(4k+6)(k+1)(k+2)(k+3) - \frac{1}{2}k(k+1)(k+2)(k+3) \\
= & (k+1)(k+2)(k+3) \left[\frac{2k+5}{4} - \frac{1}{4} + (1+w) - 3(k+2) \right. \\
+ & \left. \left(-\frac{1}{2} - w\right) + \frac{3}{4}(4k+6) - \frac{1}{2}k \right] \\
= & (k+1)(k+2)(k+3) * 0 = 0
\end{aligned}$$

The coefficient of $\sum_{i=0}^w \binom{i}{k+2} \Delta_i$ follows from $\binom{i}{k+2} i^2$, $\binom{i}{k+2} i$, $\binom{i}{k+1} i$, $\binom{i}{k+1} i^2$, $\binom{i}{k+1} i^3$,

$\binom{i}{k}i^2, \binom{i}{k}i^3, \binom{i}{k}i^4, \binom{i}{k-1}i^3, \binom{i}{k-1}i^4, \binom{i}{k-2}i^4$, therefore it equals to:

$$\begin{aligned}
& \frac{(k+1)(k+2)}{4}(k+2)^2 - \frac{(k+1)(k+2)}{4}(k+2) + (k+1)(1-w)(k+2) \\
& + (k+1)(1+w)(k+2)(2k+3) - (k+1)(k+2)(3(k+1))^2 \\
& + 3(k+1) + 1 + \left(-\frac{5}{4} + w\right)(k+1)(k+2) + \left(-\frac{1}{2} - w\right)3(k+1)^2(k+2) \\
& + \frac{3}{4}(k+1)(k+2)(6k^2 + 12k + 7) + wk(k+1)(k+2) \\
& - \frac{1}{2}(4(k-1) + 6)k(k+1)(k+2) + \frac{1}{4}(k-1)k(k+1)(k+2) \\
& = (k+1)(k+2) \left[\frac{(k+2)^2}{4} - \frac{k+2}{4} + (1-w) + (1+w)(2k+3) \right. \\
& - (3(k+1)^2 + 3k + 4) + \left(-\frac{5}{4} + w\right)\left(-\frac{1}{2} - w\right)3(k+1) \\
& + \left. \frac{3}{4}(6k^2 + 12k + 7) + wk - \frac{1}{2}(4k+2)k + \frac{1}{4}(k-1)k \right] \\
& = (k+1)(k+2) * 0 = 0
\end{aligned}$$

The coefficient of $\sum_{i=0}^w \binom{i}{k+1} \Delta_i$ follows from $\binom{i}{k+1}i, \binom{i}{k+1}i^2, \binom{i}{k+1}i^3, \binom{i}{k}i^2, \binom{i}{k}i^3, \binom{i}{k}i^4, \binom{i}{k-1}i^2, \binom{i}{k-1}i^3, \binom{i}{k-1}i^4, \binom{i}{k-2}i^3, \binom{i}{k-2}i^4$, therefore it equals to:

$$\begin{aligned}
& (k+1)(1-w)(k+1) + (k+1)(1+w)(k+1)^2 - (k+1)(k+1)^3 \\
& + \left(-\frac{5}{4} + w\right)(k+1)(2k+1) + \left(-\frac{1}{2} - w\right)(k+1)(3k^2 + 3k + 1) \\
& + \left(\frac{3}{4}(k+1)(4k^3 + 6k^2 + 4k + 1) + \left(\frac{3}{2} - w\right)k(k+1) + w3k^2(k+1)\right. \\
& - \left.\frac{1}{2}k(k+1)(6(k-1)^2 + 12(k-1) + 7)\right) \\
& + \frac{(2-4w)}{4}(k-1)k(k+1) + \frac{1}{4}(4(k-2) + 6)(k-1)k(k+1) = 0
\end{aligned}$$

The coefficient of $\sum_{i=0}^w \binom{i}{k} \Delta_i$ follows from $\binom{i}{k}, \binom{i}{k}i^2, \binom{i}{k}i^3, \binom{i}{k}i^4, \binom{i}{k-1}i, \binom{i}{k-1}i^2, \binom{i}{k-1}i^3,$

$\binom{i}{k-1}i^4, \binom{i}{k-2}i^2, \binom{i}{k-2}i^3, \binom{i}{k-2}i^4$, therefore it equals to:

$$\begin{aligned}
& (1 + \frac{5w^2}{4} - \frac{w^3}{2} + \frac{w^4}{4}) + (-\frac{5}{4} + w)k^2 + (-\frac{1}{2} - w)k^3 + \frac{3}{4}k^4 + (-3w + 2w^2 - w^3)k \\
& + (\frac{3}{2} - w)k(2(k-1) + 1) + wk(3(k-1)^2 + 3(k-1) + 1) \\
& - \frac{1}{2}k(4(k-1)^3 + 6(k-1)^2 + 4(k-1) + 1) + \frac{(1 - 6w + 6w^2)}{4}(k-1)k \\
& + \frac{(2 - 4w)}{4}3(k-1)^2k + \frac{1}{4}(k-1)k(6(k-2)^2 + 12(k-2) + 7) \\
& = \frac{1}{4}(4 + k^4 + 5w^2 - 2w^3 + w^4 - 2k^3(1 + 2w) + k^2(7 + 2w + 6w^2) \\
& - 2k(3 + 5w - w^2 + 2w^3))
\end{aligned}$$

Now we calculate the two remaining coefficients:

The coefficient of $\sum_{i=0}^w \binom{i}{k-1}\Delta_i$ follows from $\binom{i}{k-1}, \binom{i}{k-1}i, \binom{i}{k-1}i^2, \binom{i}{k-1}i^3, \binom{i}{k-1}i^4, \binom{i}{k-2}i, \binom{i}{k-2}i^2, \binom{i}{k-2}i^3, \binom{i}{k-2}i^4$, therefore it equals to:

$$\begin{aligned}
& (\frac{3}{2}w^2 - w^3 + \frac{w^4}{2}) + (-3w + 2w^2 - w^3)(k-1) + (\frac{3}{2} - w)(k-1)^2 + w(k-1)^3 \\
& - \frac{1}{2}(k-1)^4 + \frac{1}{4}(-2w + 6w^2 - 4w^3)(k-1) \\
& + \frac{1}{4}(1 - 6w + 6w^2)(k-1)(2(k-2) + 1) + \frac{1}{4}(2 - 4w)(k-1)(3(k-2)^2 \\
& + 3(k-2) + 1) + \frac{1}{4}(k-1)(4(k-2)^3 + 6(k-2)^2 + 4(k-2) + 1) \\
& = \frac{1}{2}(1 - k + w)^2(4 + k^2 + w^2 - 2k(1 + w))
\end{aligned}$$

The coefficient of $\sum_{i=0}^w \binom{i}{k-2}\Delta_i$ follows from $\binom{i}{k-2}, \binom{i}{k-2}i, \binom{i}{k-2}i^2, \binom{i}{k-2}i^3, \binom{i}{k-2}i^4$, therefore it equals to:

$$\begin{aligned}
& \frac{1}{4}(w^2 - 2w^3 + w^4) + \frac{1}{4}(-2w + 6w^2 - 4w^3)(k-2) + \frac{1}{4}(1 - 6w + 6w^2)(k-2)^2 \\
& + \frac{1}{4}(2 - 4w)(k-3)^3 + (k-2)^4 \\
& = \frac{1}{4}(1 - k + w)^2(2 - k + w)^2
\end{aligned}$$

Finally, we get the following formula:

$$\begin{aligned}
0 &= \frac{1}{4}[4 + k^4 + 5w^2 - 2w^3 + w^4 - 2k^3(1 + 2w) + k^2(7 + 2w + 6w^2) \\
&\quad - 2k(3 + 5w - w^2 + 2w^3)] \sum_{i=0}^w \binom{i}{k} \Delta_i \\
&\quad + \frac{1}{2}(1 - k + w)^2(4 + k^2 + w^2 - 2k(1 + w)) \sum_{i=0}^w \binom{i}{k-1} \Delta_i \\
&\quad + \frac{1}{4}(1 - k + w)^2(2 - k + w)^2 \sum_{i=0}^w \binom{i}{k-2} \Delta_i
\end{aligned}$$

□

Appendix C

Proof of proposition 52

Proposition 52.

$$\begin{aligned} \sum_{i=0}^w \left[\binom{w-i-1}{k-1} (w-i)^2 - \binom{w-i}{k-1} i^2 + \binom{i-1}{k-1} i^2 - \binom{i}{k-1} (w-i)^2 \right] A_i = \\ = 2(2wk - k^2 + k) \sum_{i=0}^w \binom{i}{k} A_i - 2(w - (k-1))^2 \sum_{i=0}^w \binom{i}{k-1} A_i. \end{aligned}$$

Proof. The code is self-complement by [15], thus $A_i = A_{w-i}$, therefore

$$\begin{aligned} 2w^2 \sum_{i=0}^w \binom{i}{k} \Delta_i &= \sum_{i=0}^w \binom{w-i-1}{k-1} (w-i)^2 A_{w-i} - \sum_{i=0}^w \binom{w-i}{k-1} i^2 A_{w-i} \\ &+ \sum_{i=0}^w \binom{i-1}{k-1} i^2 A_i - \sum_{i=0}^w \binom{i}{k-1} (w-i)^2 A_i \\ &= \sum_{i=0}^w \binom{i-1}{k-1} i^2 A_i - \sum_{i=0}^w \binom{i}{k-1} (w-i)^2 A_i \\ &+ \sum_{i=0}^w \binom{i-1}{k-1} i^2 A_i - \sum_{i=0}^w \binom{i}{k-1} (w-i)^2 A_i \\ &= 2 \sum_{i=0}^w \binom{i-1}{k-1} i^2 A_i - 2 \sum_{i=0}^w \binom{i}{k-1} (w-i)^2 A_i \end{aligned}$$

now we use the following equalities:

$$\binom{i-1}{k-1} i^2 = \binom{i}{k} ik$$

$$\begin{aligned}\binom{i}{k}ik - \binom{i}{k-1}i^2 &= -i(k-1)\binom{i}{k-1} \\ i\binom{i}{k-1} &= k\binom{i}{k} + (k-1)\binom{i}{k-1}\end{aligned}$$

$$\begin{aligned}w^2 \sum_{i=0}^w \binom{i}{k} \Delta_i &= \sum_{i=0}^w \binom{i}{k} ik A_i - w^2 \sum_{i=0}^w \binom{i}{k-1} A_i \\ &+ 2w \sum_{i=0}^w \binom{i}{k-1} i A_i - \sum_{i=0}^w \binom{i}{k-1} i^2 A_i \\ &= -(k-1) \sum_{i=0}^w \binom{i}{k-1} i A_i - w^2 \sum_{i=0}^w \binom{i}{k-1} A_i + 2w \sum_{i=0}^w \binom{i}{k-1} i A_i \\ &= (2w - (k-1))k \sum_{i=0}^w \binom{i}{k} A_i \\ &+ (2w - (k-1))(k-1) \sum_{i=0}^w \binom{i}{k-1} A_i - w \sum_{i=0}^w \binom{i}{k-1} A_i \\ &= (2wk - k^2 + k) \sum_{i=0}^w \binom{i}{k} A_i \\ &+ (2w(k-1) - (k-1)^2 - w^2) \sum_{i=0}^w \binom{i}{k-1} A_i \\ &= (2wk - k^2 + k) \sum_{i=0}^w \binom{i}{k} A_i - (w - (k-1))^2 \sum_{i=0}^w \binom{i}{k-1} A_i\end{aligned}$$

Finally, we get

$$w^2 \sum_{i=0}^w \binom{i}{k} \Delta_i = (2wk - k^2 + k) \sum_{i=0}^w \binom{i}{k} A_i - (w - k + 1)^2 \sum_{i=0}^w \binom{i}{k-1} A_i$$

□

Bibliography

- [1] F. J. MacWilliams and N. J. Sloane. *The Theory of Error-Correcting Codes*. Amsterdam, North-Holland, 1977.
- [2] A. Tietäväinen. On the nonexistence of perfect codes over finite fields. *SIAM J. Appl. Math.*, 24:88–96, 1973.
- [3] J. H. van Lint. Nonexistence theorems for perfect error-correcting codes. In *Computers in Algebra and Number Theory, vol. IV, SIAM-AMS Proceedings*, 1971.
- [4] V. A. Zinoviev and V. K. Leontiev. The nonexistence of perfect codes over galois fields. *Probl. Control and Info. Theory*, 2:123–132, 1973.
- [5] M. R. Best. Perfect codes hardly exists. *IEEE Trans. on Inform. Theory*, 29:349–351, 1983.
- [6] P. Delsarte. An algebraic approach to association schemes of coding theory. *Philips J. Res.*, 10:1–97, 1973.
- [7] S. M. Johnson. A new upper bound for error-correcting codes. *IRE Trans. on Inform. Theory*, 8:203–207, 1962.
- [8] A. Vardy E. Agrell and K. Zeger. Upper bounds for constant-weight codes. *IEEE Trans. on Inform. Theory*, 46:2373–2395, 2000.
- [9] E. Biggs. Perfect codes in graphs. *J. Combin. Theory Ser. B*, 15:289–296, 1973.
- [10] E. Bannai. Codes in bi-partite distance-regular graphs. *Journal London Math. Soc.*, 2:197–202, 1977.
- [11] P. Hammond. On the non-existence of perfect codes and nearly perfect codes. *Discrete Math.*, 39:105–109, 1982.

- [12] C. Roos. A note on the existence of perfect constant weight codes. *Discrete Mathematics*, 47:121–123, 1983.
- [13] T. Etzion. On perfect codes in the johnson scheme. *DIMACS series in Discrete Mathematics and Theoretical Computer Science*, 56:125–130, 2001.
- [14] T. Etzion and M. Schwartz. Perfect constant-weight codes. *IEEE Trans. on Inform. Theory*, 50:2156–2165, 2004.
- [15] T. Etzion. On the nonexistence of perfect codes in the johnson scheme. *SIAM J. Discrete Math*, 9:201–209, 1996.
- [16] N. J. A. Sloane A. E. Brouwer, J. B. Shearer and W. D. Smith. A new table of constant weight codes. *IEEE Trans. on Inform. Theory*, 36:1334–1380, 1990.
- [17] W. J. Martin. *Completely regular subsets*. Ph. D. dissertation, Univ. Waterloo, 1992.
- [18] T. Etzion. Configuration distribution and designs of codes in the johnson scheme. *Journal of Combinatorial designs*, 15:15–34, 2006.
- [19] H. K. Aydinian R. Ahlswede and L. H. Khachatrian. On perfect codes and related concepts. *Designs, Codes and Cryptography*, 22(3):221–237, 2001.
- [20] D. M. Gordon. Perfect single error-correcting codes in the Johnson scheme. *IEEE Transactions on Information Theory*, (52):4670–4672, 2006.
- [21] V. Pless. Power moments identities on weight distributions in error correcting codes. *Information and Control*, 6:147–152, 1963.
- [22] V. Pless. *Introduction to the Theory of Error-Correcting Codes*. Wiley, New York, 1990.
- [23] E. W. Weisstein. "pell equation". from mathworld – a wolfram web resource. <http://mathworld.wolfram.com/PellEquation.html>.
- [24] R Stanley and E. W. Weisstein. "catalan number." from mathworld–a wolfram web resource. <http://mathworld.wolfram.com/CatalanNumber.html>.
- [25] A. Vardy E. Agrell and K. Zeger. Tables of binary block codes, [online]. <http://www.s2.chalmers.se/argell>.
- [26] T. Etzion. Optimal doubly constant weight codes, preprint.

- [27] D. Jungnickel Th. Beth and H. Lenz. *Design Theory*. Cambridge University Press, Cambridge, 1986. 123 pp.